

Coclass theory for nilpotent associative algebras

Von der
Carl-Friedrich-Gauß-Fakultät
der Technischen Universität Carolo-Wilhelmina zu Braunschweig

zur Erlangung des Grades eines
Doktors der Naturwissenschaften (Dr. rer. nat.)

genehmigte Dissertation

von
Tobias Moede
geboren am 13.09.1987
in Hamburg

Eingereicht am: 02.05.2016
Disputation am: 01.07.2016
1. Referentin: Prof. Dr. Bettina Eick
2. Referent: Prof. Dr. Andrea Caranti

Contents

1	Introduction	5
2	Preliminaries	13
3	Infinite paths in coclass graphs	17
3.1	Description of infinite paths	17
3.2	Construction of infinite paths	25
3.3	Number of infinite paths	31
4	Trees in coclass graphs	37
4.1	Roots of maximal descendant trees	37
4.2	Depth of coclass trees	41
4.3	General periodicity conjecture	46
4.4	Periodicity for coclass 1	47
4.5	Periodicity conjecture for coclass 2	51
5	Algorithms	53
5.1	Algorithm to compute immediate descendants	53
5.2	Algorithm to calculate roots of maximal descendant trees	55
6	Applications of the algorithms	57
6.1	The coclass graph $\mathcal{G}_{\mathbb{F}}(1)$ with $ \mathbb{F} < 20$	57
6.2	The coclass graph $\mathcal{G}_{\mathbb{F}}(2)$ with $ \mathbb{F} \leq 5$	59
6.3	The coclass graph $\mathcal{G}_{\mathbb{F}_2}(3)$	68
7	Summary	75
7.1	Summary	75
7.2	Zusammenfassung	78
	Bibliography	83

Chapter 1

Introduction

Coclass theory can be considered as an approach towards classifying nilpotent algebraic objects. In this thesis we develop a coclass theory for nilpotent associative algebras over fields. The thesis contains results from our papers [EM15b] & [EM16] as well as various new results.

Historically coclass theory was first introduced for finite p -groups by Leedham-Green & Newman [LGN80] in 1980. Ideas of coclass theory have been translated to special types of Lie algebras, see e.g. Caranti, Mattarei & Newman [CMN97], Caranti & Vaughan-Lee [CVL00], Riley & Semple [RS94] and Shalev & Zelmanov [SZ97]. Ideas have also been transferred to nilpotent semigroups by Distler & Eick [DE13]. We first recall some of the highlights of coclass theory for finite p -groups before we give an overview of the main results of this thesis.

Coclass theory for finite p -groups

A group G of order p^n and nilpotency class c has **coclass** $cc(G) = n - c$. Inspired by Blackburn's work [Bla58] on groups of maximal class, i.e. coclass 1, Leedham-Green & Newman suggested to use coclass as primary invariant in a possible classification of finite p -groups. In [LGN80] Leedham-Green & Newman proved first results and proposed five very detailed conjectures known as Conjectures A-E. These conjectures have become a driving force for the first years of coclass theory. We exhibit the conjectures here for completeness and refer to the monograph by Leedham-Green & McKay [LGM02] for details. The conjectures involve infinite pro- p -groups of coclass r , i.e. infinite groups that are inverse limits of finite p -groups of coclass r . Conjectures A-E are ordered in decreasing strength, i.e. Conjecture A implies all the others.

Conjecture A. For some function $f(p, r)$, every finite p -group of coclass r has a normal subgroup K of class at most 2 and index at most $f(p, r)$. If $p = 2$, one can require K to be abelian.

Conjecture B. For some function $g(p, r)$, every finite p -group of coclass r has derived length at most $g(p, r)$.

Conjecture C. Every pro- p -group of finite coclass is soluble.

Conjecture D. For fixed p and r there are only finitely many isomorphism classes of infinite pro- p -groups of coclass r .

Conjecture E. There are only finitely many isomorphism classes of infinite soluble pro- p -groups of coclass r .

The work on these conjectures has produced many interesting results. All of these conjectures have become theorems by now, hence we refer to them as Theorems A-E. The strongest one, Theorem A, has been proved by Leedham-Green [LG94] and Shalev [Sha94].

One can associate to the p -groups of a fixed coclass r a directed graph $\mathcal{G}(p, r)$. The vertices correspond one-to-one to isomorphism types of finite p -groups of coclass r and there is a directed edge from H to G if $G/\gamma(G) \cong H$, where $\gamma(G)$ is the last non-trivial term of the lower central series of H . Newman and O'Brien [NO99] conjectured that the graphs $\mathcal{G}(2, r)$ exhibit a certain periodic pattern (**Conjecture P**). This was proved independently by du Sautoy [dS00] using the theory of zeta functions and Eick & Leedham-Green [ELG08] using cohomological methods.

It seems that coclass theory for finite p -groups is very helpful to describe global structure phenomena. Eick [Eic06] showed that for 2-groups the automorphism groups also exhibit a periodic pattern. Eick & Leedham-Green [ELG08] introduced so called infinite coclass sequences and in [EF11] Eick & Feichtenschlager proved that low-dimensional cohomology groups agree for almost all groups in an infinite coclass sequence. Couson [Cou14] showed that for an infinite coclass sequence there is an integer b , such that every irreducible character of every group in the sequence has degree at most p^b . These and other structure results are part of the reason why coclass theory for finite p -groups is considered as a successful approach.

There is still active work on coclass theory for finite p -groups. The most prominent and important open question is whether for fixed p and r , the p -groups of coclass r can be described by finitely many parametrized presentations. This is related to **Conjecture W** by Eick, Leedham-Green, Newman & O'Brien [ELGNO13]. In the case of 2-groups this is implied by the periodicity result obtained by Eick & Leedham-Green [ELG08].

Nilpotent associative \mathbb{F} -algebras

An **associative algebra** over a field \mathbb{F} is an \mathbb{F} -vector space equipped with an associative, bilinear multiplication. If an associative \mathbb{F} -algebra contains an identity element, we call it **unital**, otherwise **non-unital**. Associative algebras are found in many areas of mathematics. Examples include the algebra $M_n(\mathbb{F})$ of $(n \times n)$ -matrices with entries in \mathbb{F} and its various subalgebras, group algebras $\mathbb{F}G$ for finite groups G and many others. An important example for our purpose is the ideal $\mathbb{F}_o[[t]]$ generated by t in the ring of formal power series $\mathbb{F}[[t]]$. It is a non-unital infinite-dimensional associative \mathbb{F} -algebra.

Definition 1.1

Let A be an associative \mathbb{F} -algebra and let A^i denote the ideal generated by all products of length i in A . We call the series

$$A = A^1 > A^2 > \dots$$

the **power series** of A . We say that A is **nilpotent** if there is $c \in \mathbb{N}$, such that $A^c \neq \{0\}$ and $A^{c+1} = \{0\}$. In this case we call c the **class** of A and denote it by $cl(A)$.

Note that nilpotent associative \mathbb{F} -algebras are always non-unital. The importance of nilpotent associative \mathbb{F} -algebras in the structure theory of finite-dimensional associative \mathbb{F} -algebras is illustrated by a theorem of Wedderburn (see [Wed08]), which states that the Jacobson radical $J(A)$ of a finite-dimensional associative \mathbb{F} -algebra A is a nilpotent associative \mathbb{F} -algebra and $A/J(A)$ is a direct sum of full matrix algebras over skewfields.

An isomorphism type classification of the nilpotent associative \mathbb{F} -algebras seems to be an interesting project. The classification is available for small dimensions. It is easy to see that there is one such algebra of dimension 1: $C_1 = \mathcal{Z}_1 = \langle a \mid a^2 \rangle$, and there are two such algebras of dimension 2: $\mathcal{Z}_2 = \mathcal{Z}_1 \oplus \mathcal{Z}_1$ and $C_2 = \langle a \mid a^3 \rangle$. The number of nilpotent associative \mathbb{F} -algebras of dimension 3 depends on the underlying field (see de Graaf [dG10], Kruse & Price [KP69]). If \mathbb{F} is infinite, then there are infinitely many such algebras. If \mathbb{F} is finite, then there are $|\mathbb{F}| + 6$ such algebras if $\text{char}(\mathbb{F})$ is odd and $|\mathbb{F}| + 5$ if $\text{char}(\mathbb{F})$ is even. Kruse & Price give a classification of nilpotent associative \mathbb{F} -algebras of dimension 4 (see [KP69, Chapter VI]).

Coclass theory for nilpotent associative \mathbb{F} -algebras

In this thesis we take a different classification approach: coclass theory. In analogy to the case of finite p -groups we define the coclass of a finite-dimensional nilpotent-associative \mathbb{F} -algebra.

Definition 1.2

Let A be a finite-dimensional nilpotent associative \mathbb{F} -algebra. We define the **coclass** of A as $cc(A) = \dim(A) - cl(A)$.

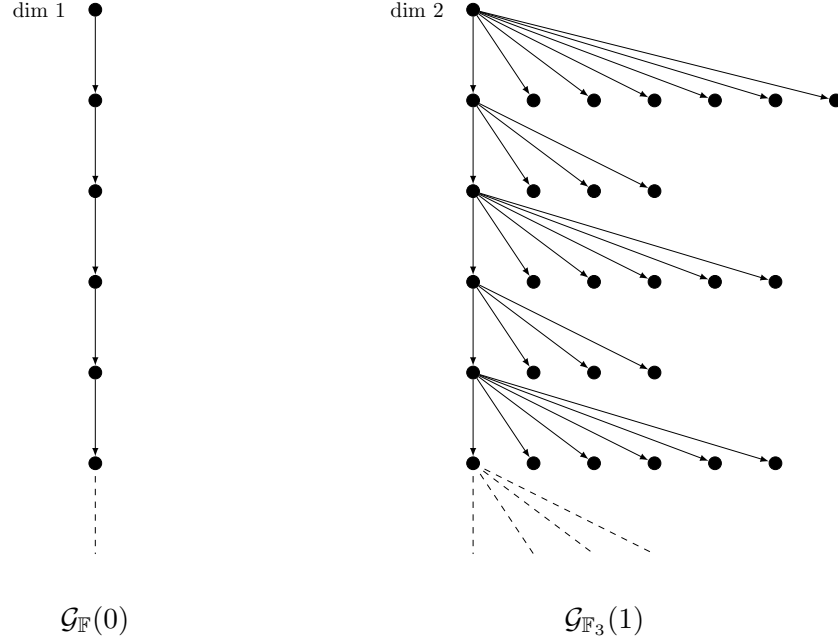
A very important tool in the investigation of finite-dimensional nilpotent associative \mathbb{F} -algebras of coclass r are the associated coclass graphs, which we define in the following. Note that we exhibit some more terminology and some elementary facts in Chapter 2.

Let $A \not\cong B$ be two finite-dimensional nilpotent associative \mathbb{F} -algebras. We call B a **descendant** of A (and A an **ancestor** of B), if $B/B^i \cong A$ for some $i > 1$. In the special case that $B/B^{cl(B)} \cong A$, we say that B is an **immediate descendant** of A and similarly that A is an **immediate ancestor** of B .

Definition 1.3

Let \mathbb{F} be an arbitrary field and let r be a non-negative integer. We define the **coclass graph** $\mathcal{G}_{\mathbb{F}}(r)$ as follows. The vertices correspond one-to-one to the isomorphism types of finite-dimensional nilpotent associative \mathbb{F} -algebras of coclass r and there is a directed edge between A and B if B is an immediate descendant of A .

Note that we do often not distinguish between the vertex and an algebra representing the isomorphism type that corresponds to it. As first examples we depict the coclass graph for coclass 0 and all fields and the coclass graph for coclass 1 and the field with three elements.



Both of these graphs have been calculated up to dimension 40 using the algorithms we present in Chapter 5. Both graphs exhibit a periodic pattern, which we investigate in more detail in Chapter 4. Note that the graphs for higher coclasses get significantly more complex.

Infinite paths in coclass graphs

We observe that a main feature of coclass graphs are the infinite paths contained in them. We use inverse limits as a convenient and compact description for them. A central result of this thesis (Theorem 1 in our paper [EM15b]) is the following structure description for the inverse limits associated to infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$. Note that $\text{Ann}_*(A)$ is the direct limit of the upper annihilator series of A (see Definition 3.13). We give more details and a proof in Section 3.1.

Theorem 1.4

Let \mathbb{F} be an arbitrary field and let r be a non-negative integer. An associative \mathbb{F} -algebra A is isomorphic to an inverse limit of an infinite path in $\mathcal{G}_{\mathbb{F}}(r)$ if and only if it is a split extension $A = T \ltimes \text{Ann}_(A)$ with $\dim(\text{Ann}_*(A)) = r$ and T is a subalgebra of A such that $T \cong \mathbb{F}_o[[t]]$.*

This can be seen as an analogue to Conjecture C from the p -group case. We call two infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$ **equivalent** if one is contained in the other. In Section 3.3 we further deduce that an analogue of Conjecture D holds if and only if $r \leq 1$ or \mathbb{F} is finite, i.e. we have the following theorem. Note that this result is also contained in Section 5 of our paper [EM15b], but we use different methods to prove it here.

Theorem 1.5

The number of equivalence classes of infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$ is finite if and only if $r \leq 1$ or \mathbb{F} is finite.

In Section 3.2 we explicitly describe the construction of the inverse limits describing infinite paths in coclass graphs as so called iterated annihilator extensions of $\mathbb{F}_o[[t]]$. We solve the arising isomorphism problem using an action of automorphism groups on cohomology, in particular see Lemma 3.32 and Theorem 3.33.

Trees in coclass graphs

Let \mathbb{F} be an arbitrary field and let r be a non-negative integer and let $A \in \mathcal{G}_{\mathbb{F}}(r)$. The **descendant tree** $\mathcal{T}(A)$ of A is the full subtree of $\mathcal{G}_{\mathbb{F}}(r)$ consisting of A and all descendants of A in $\mathcal{G}_{\mathbb{F}}(r)$. It is a rooted tree with root A . $\mathcal{T}(A)$ is called **maximal** if it is not properly contained in the descendant tree of an ancestor of A . The descendant tree $\mathcal{T}(A)$ can be finite or infinite with one or several infinite paths starting at A .

In Section 4.1 we describe the shape of the coclass graphs $\mathcal{G}_{\mathbb{F}}(r)$ in more detail. These results are also contained in our paper [EM16].

Theorem 1.6

Let \mathbb{F} be an arbitrary field and let r be a non-negative integer. Let R be a root of a maximal descendant tree in $\mathcal{G}_{\mathbb{F}}(r)$. If $r = 0$, then $\dim(R) = 1$. If $r \geq 1$, then $r + 1 \leq \dim(R) \leq 2r$ and these bounds are sharp.

Corollary 1.7

Let \mathbb{F} be an arbitrary finite field and let r be a non-negative integer. Then $\mathcal{G}_{\mathbb{F}}(r)$ is a disjoint union of finitely many maximal descendant trees.

This result can be further strengthened using the notion of coclass trees.

Definition 1.8

*Let \mathbb{F} be an arbitrary field and let r be a non-negative integer. Let $A \in \mathcal{G}_{\mathbb{F}}(r)$. A descendant tree $\mathcal{T}(A)$ in $\mathcal{G}_{\mathbb{F}}(r)$ is called a **coclass tree** if it contains a unique infinite path starting at its root. A coclass tree $\mathcal{T}(A)$ is called **maximal** if it is not properly contained in another coclass tree.*

Theorem 1.9

Let \mathbb{F} be an arbitrary finite field and let r be a non-negative integer. Then each maximal descendant tree of $\mathcal{G}_{\mathbb{F}}(r)$ consists of finitely many maximal coclass trees and finitely many other vertices. Hence $\mathcal{G}_{\mathbb{F}}(r)$ is a disjoint union of finitely many maximal coclass trees and finitely many other vertices.

Definition 1.10

Let \mathbb{F} be an arbitrary field and let r be a non-negative integer. Let \mathcal{T} be a maximal coclass tree in $\mathcal{G}_{\mathbb{F}}(r)$ and let $A_1 \rightarrow A_2 \rightarrow \dots$ be the unique infinite path starting at the root of \mathcal{T} .

- *The subtree \mathcal{B}_i of $\mathcal{T}(A_i)$ containing all algebras that are not contained in $\mathcal{T}(A_{i+1})$ is called the **i -th branch** of \mathcal{T} . The branch \mathcal{B}_i is a finite tree with root A_i .*
- *We define the **depth** $\text{dep}(\mathcal{B}_i)$ of \mathcal{B}_i to be the maximum distance of a vertex in \mathcal{B}_i to the algebra A_i . If the sequence $\text{dep}(\mathcal{B}_i)_{i \in \mathbb{N}}$ is bounded, we define the **depth** of \mathcal{T} as $\text{dep}(\mathcal{T}) = \max_{i \in \mathbb{N}} \text{dep}(\mathcal{B}_i)$. In this case we say that \mathcal{T} has **bounded depth**. Otherwise \mathcal{T} has **unbounded depth** and we write $\text{dep}(\mathcal{T}) = \infty$.*

We prove in Section 4.2 the new result that for almost all algebras A in a maximal coclass tree in $\mathcal{G}_{\mathbb{F}}(r)$ the algebra $A/A^{cl(A)-r+1}$ is isomorphic to an algebra on the infinite path. In other words, we have that almost all branches of maximal coclass trees in $\mathcal{G}_{\mathbb{F}}(r)$ have depth bounded by the coclass and hence we have the following theorem.

Theorem 1.11

Let \mathbb{F} be an arbitrary field and let r be a non-negative integer. Let \mathcal{T} be a maximal coclass tree in $\mathcal{G}_{\mathbb{F}}(r)$. Then \mathcal{T} has bounded depth.

Periodicity of coclass graphs

For finite fields \mathbb{F} , we propose strong conjectures in Section 4.3 resembling Conjecture P and the consequences of Conjecture W from the p -group case (see also our paper [EM16]). These conjectures are based on computational evidence that we obtained using the algorithms we present in Chapter 5. A preliminary version of the descendant algorithm (see Algorithm 5.1) was used by Bertram [Ber11] to calculate the 5-dimensional nilpotent associative \mathbb{F}_2 -algebras.

Definition 1.12

*Let \mathbb{F} be an arbitrary field and let r be a non-negative integer. Let \mathcal{T} be a maximal coclass tree in $\mathcal{G}_{\mathbb{F}}(r)$ with unique infinite path $A_1 \rightarrow A_2 \rightarrow \dots$ starting at the root of \mathcal{T} . Then \mathcal{T} is called **virtually periodic** with **period** d and **periodic root** A_ℓ if the descendant trees $\mathcal{T}(A_i)$ and $\mathcal{T}(A_{i+d})$ are isomorphic as directed graphs for each $i \geq \ell$.*

Conjecture 1.13

Let \mathbb{F} be an arbitrary finite field and let r be a non-negative integer and let \mathcal{T} be maximal coclass tree in $\mathcal{G}_{\mathbb{F}}(r)$. Then \mathcal{T} is virtually periodic.

Conjecture 1.14

Let \mathbb{F} be an arbitrary finite field and let r be a non-negative integer and let \mathcal{T} be maximal coclass tree in $\mathcal{G}_{\mathbb{F}}(r)$. Then the algebras in \mathcal{T} can be described by finitely many parametrized presentations.

In Section 4.4 we give a first proof of these conjectures in the case of coclass $r \leq 1$ and for coclass 2 give a far more detailed conjecture in Section 4.5.

Outline of the thesis

In the following we outline the structure of the thesis.

In Chapter 2 we introduce more basic definitions and notions of coclass theory for nilpotent associative \mathbb{F} -algebras.

In Chapter 3 we deal with the infinite paths of coclass graphs associated to the nilpotent associative \mathbb{F} -algebras of a fixed coclass r . It contains the central structure description for the inverse limits describing those infinite paths as well as a method for explicitly constructing these inverse limits. The number of infinite paths in coclass graphs is also investigated here.

In Chapter 4 we consider trees in coclass graphs. We prove a bound for the dimension of roots of maximal descendant trees in coclass graphs and prove a depth bound for coclass

trees. Furthermore, we make strong conjectures involving periodicity of coclass graphs for finite fields and prove these conjectures for coclass 1.

In Chapter 5 we describe algorithms that enable us to calculate finite parts of coclass graphs for finite fields.

In Chapter 6 we apply the algorithms of the previous chapter. As a first application we fully describe the graphs $\mathcal{G}_{\mathbb{F}}(1)$ for all finite fields of size less than 20. We also depict the graph $\mathcal{G}_{\mathbb{F}}(2)$ for finite fields with $|\mathbb{F}| \leq 5$ and the graph $\mathcal{G}_{\mathbb{F}_2}(3)$.

In Chapter 7 we give a summary of what has been achieved in this thesis.

Chapter 2

Preliminaries

Let \mathbb{F} be an arbitrary field. We introduce more terminology needed to develop a coclass theory for nilpotent associative \mathbb{F} -algebras. We state and prove small helpful facts and give some examples.

Definition 2.1

Recall from Definition 1.1 that an associative \mathbb{F} -algebra is nilpotent of class c if its power series has the form $A = A^1 > A^2 > \dots > A^c > A^{c+1} = \{0\}$. We call $rk(A) = \dim(A/A^2)$ the **rank** of A .

Remark 2.2

The rank of a nilpotent associative \mathbb{F} -algebra A coincides with its minimal generator number.

Of course it would be desirable to have a classification of the nilpotent associative \mathbb{F} -algebras up to isomorphism by dimension, but as discussed in the introduction, this seems to be a hard and daunting task. Thus we settle for a different approach, namely the use of coclass as the primary invariant in a classification attempt. Recall from Definition 1.2 that the coclass of a nilpotent associative \mathbb{F} -algebra is defined as $\dim(A) - cl(A)$.

Lemma 2.3

A nilpotent associative \mathbb{F} -algebra A has coclass r if and only if

$$\sum_{i=1}^{cl(A)} (\dim(A^i/A^{i+1}) - 1) = r$$

Proof:

$$cc(A) = \dim(A) - cl(A) = \sum_{i=1}^{cl(A)} \dim(A^i/A^{i+1}) - cl(A) = \sum_{i=1}^{cl(A)} (\dim(A^i/A^{i+1}) - 1). \quad \square$$

We proceed by making some elementary observations concerning rank and coclass, in particular we see that the rank is bounded in terms of the coclass.

Lemma 2.4

Let A be a finite-dimensional nilpotent associative \mathbb{F} -algebra. Then

(a) $cc(A) \geq 0$, and

- (b) $rk(A) \leq cc(A) + 1$.

Proof:

- (a) Obviously $cl(A) \leq \dim(A)$, hence $cc(A) \geq 0$.
 (b) $rk(A) - 1 \leq cc(A/A^2) \leq cc(A/A^3) \leq \dots \leq cc(A/A^{cl(A)+1}) = cc(A)$.

□

We first give a few examples of finite-dimensional nilpotent associative \mathbb{F} -algebras.

Example 2.5

Let \mathbb{F} be an arbitrary field.

- (a) Consider the finite-dimensional nilpotent associative \mathbb{F} -algebras given by $C_n = \langle a \mid a^{n+1} \rangle$ for each $n \geq 1$. These rank 1 algebras are n -dimensional and have class n . Thus they are of coclass 0. In fact the algebras C_n are all the nilpotent associative \mathbb{F} -algebras of coclass 0. They (and their inverse limit) play an important role in the theory. Note that in the p -group case coclass 0 is non-existent.
- (b) We denote the d -dimensional \mathbb{F} -algebras with trivial multiplication by \mathcal{Z}_d . These have class 1 and hence coclass $d - 1$ meaning that there are nilpotent associative \mathbb{F} -algebras for every coclass $r \geq 0$. The algebras C_n for coclass 0 and $C_n \oplus \mathcal{Z}_d$ for coclass $d \geq 1$ show that there are infinitely many nilpotent associative \mathbb{F} -algebras for every coclass $r \geq 0$.
- (c) Let $SUT(n; \mathbb{F})$ be the \mathbb{F} -algebra of strictly upper triangular $(n \times n)$ -matrices over \mathbb{F} . Then $SUT(n; \mathbb{F})$ is nilpotent of dimension $n(n - 1)/2$ and class $n - 1$. Consequently, these algebras have coclass $(n - 1)(n - 2)/2$. Note that $SUT(n; \mathbb{F})$ is precisely the Jacobson radical of the \mathbb{F} -algebra of upper triangular $(n \times n)$ -matrices over \mathbb{F} .

For finite fields \mathbb{F} an interesting way of constructing nilpotent associative \mathbb{F} -algebras from finite p -groups is described in the next example. This is due to Jennings [Jen41] and a more detailed exposition of this construction can be found in Chapter VIII §2 of [HB12].

Example 2.6

Let G be a finite p -group of order p^n . Let \mathbb{F} be a finite field of characteristic p and let $A = J(\mathbb{F}G)$ be the Jacobson radical of the group algebra $\mathbb{F}G$. Furthermore, let κ_* denote the Jennings series of G and let ℓ be the length of this series. Then A is a nilpotent associative algebra of dimension $p^n - 1$ and class $(p - 1) \sum_{n=1}^{\ell} nd_n$, where d_n is defined by $|\kappa_n(G) : \kappa_{n+1}(G)| = p^{d_n}$.

Remark 2.7

Not all nilpotent associative \mathbb{F} -algebras arise as a Jacobson radical in the sense of Example 2.6. Even for the algebras that do arise in this way the fact that the Jennings series is involved in the calculation of the class shows that there is no direct way to exploit the connection for coclass theoretic results.

There is also a construction going the other direction, i.e. starting with a nilpotent associative \mathbb{F}_p -algebra and yielding a finite p -group.

Example 2.8

Let A be a finite-dimensional nilpotent \mathbb{F}_p -algebra. Then A is a p -group of order $p^{\dim(A)}$ with the composition given by

$$a \circ b = a + b + ab \quad \forall a, b \in A.$$

Remark 2.9

The composition defined in Definition 2.8 is called the **circle product** and the group (A, \circ) is called the **circle group** or **adjoint group** of A . Furthermore, the power series of A is a central series for (A, \circ) (see 1.6.5 in Kruse & Price [KP69]), thus giving an upper bound for the class of (A, \circ) . Not all finite p -groups arise in this way, e.g. the dihedral group on 16 elements is not the circle group of a nilpotent associative \mathbb{F}_p -algebra. This can be deduced from Corollary 1.6.9 in [KP69].

Remark 2.10

Kruse & Price give a slightly more general version of this construction in [KP69, Theorem 1.6.2]. More precisely they introduce this construction for nil rings, i.e. rings in which every element is nilpotent. In the general setting this construction does of course not necessarily yield p -groups.

To end the chapter we give some more terminology to precisely describe coclass graphs and prove some elementary facts.

Definition 2.11

- The distance $\text{dist}(B)$ of the descendant B is defined as $\text{dist}(B) = \text{cl}(B) - \text{cl}(A)$ and the stepsize $s(B)$ is defined as $s(B) = \dim(B) - \dim(A)$. Using this terminology the immediate descendants are the descendants of distance 1.
- We call a finite-dimensional nilpotent associative \mathbb{F} -algebra A **capable** if it has immediate descendants. Otherwise it is called **terminal**.

We have the following elementary results for rank and coclass of an immediate descendant.

Lemma 2.12

Let A be a finite-dimensional nilpotent associative \mathbb{F} -algebra and let B be an immediate descendant of A with stepsize $s(B)$. By definition we have that $s(B) \geq 1$, $\text{cl}(B) = \text{cl}(A) + 1$ and $\dim(B) = \dim(A) + s(B)$. Furthermore, the following holds:

- (a) $\text{rk}(B) = \text{rk}(A)$,
- (b) $\text{cc}(B) = \text{cc}(A) + s(B) - 1$. In particular $\text{cc}(B) \geq \text{cc}(A)$ and equality holds if and only if B has stepsize 1.

Proof:

- (a) Note that $\text{cl}(B) > 2$ since by definition we have $\text{cl}(B) > \text{cl}(A) \geq 1$, hence $B/B^2 \cong (B/B^{\text{cl}(B)})/(B^2/B^{\text{cl}(B)}) \cong A/A^2$. Thus of course $\text{rk}(B) = \text{rk}(A)$.
- (b) $\text{cc}(B) = \dim(B) - \text{cl}(B) = \dim(A) + s(B) - (\text{cl}(A) + 1) = \text{cc}(A) + s(B) - 1$.

□

In the next chapter we turn to the theoretical investigation of these graphs. The first goal is a detailed investigation of the infinite paths contained in these graphs.

Infinite paths in coclass graphs

If not explicitly mentioned, let \mathbb{F} be an arbitrary field and r a non-negative integer throughout this chapter. The results overlap with our paper [EM15b].

3.1 Description of infinite paths

In this section we investigate one of the central features of the coclass graph $\mathcal{G}_{\mathbb{F}}(r)$, namely the infinite paths contained in these graphs. Our main result can be regarded as an analogue of Theorem C in the p -group case (see Section 1).

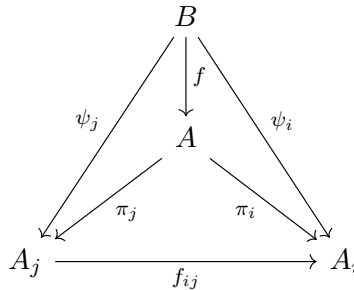
We need the concept of inverse limits to give a convenient description for the infinite paths. The general concept is defined using a universal property, which we shortly recall here (for more details on limits see e.g. Lane [Lan71, Chapter III]).

Definition 3.1

Let (I, \leq) be a directed set and let $(A_i)_{i \in I}$ be a family of objects in a category \mathcal{C} . Further suppose that there are morphisms $f_{ij} : A_j \rightarrow A_i$ for all $i \leq j$ that satisfy

- f_{ii} is the identity on A_i
- $f_{ik} = f_{ij} \circ f_{jk}$ for $i \leq j \leq k$.

Then $((A_i)_{i \in I}, (f_{ij})_{i \leq j \in I})$ is called an **inverse system**. For convenience we often just write (A_i, f_{ij}) . If we have such an inverse system (A_i, f_{ij}) , the **inverse limit** of this system is an object $A \in \mathcal{C}$ together with morphisms $\pi_i : A \rightarrow A_i$ satisfying $\pi_i = f_{ij} \circ \pi_j$ for all $i \leq j$. Additionally (A, π_i) must be universal, i.e. for every other such (B, ψ_i) there must be a unique morphism $f : B \rightarrow A$ such that the following diagram commutes for all $i \leq j$.



We often denote the inverse limit by $A = \varprojlim A_i$.

In the special case of associative algebras one can give an explicit construction for the inverse limits, which usually suffices for our considerations.

Remark 3.2

Let

$$A_1 \xleftarrow{\varphi_1} A_2 \xleftarrow{\varphi_2} \dots$$

be an infinite sequence of associative \mathbb{F} -algebras with epimorphisms $\varphi_i : A_{i+1} \rightarrow A_i$ for all $i \in \mathbb{N}$. Then the inverse limit of this sequence is defined as

$$\varprojlim A_i = \{(a_1, a_2, \dots) \in \prod_{i \in \mathbb{N}} A_i \mid \varphi_i(a_{i+1}) = a_i \text{ for all } i \in \mathbb{N}\}.$$

Example 3.3

Consider the nilpotent associative \mathbb{F} -algebras of coclass 0 given by $C_i = \langle a \mid a^{i+1} \rangle$ for $i \in \mathbb{N}$. Let $\varphi_i : C_{i+1} \rightarrow C_i$ be the natural projection that maps $a \in C_{i+1}$ to $a \in C_i$. Let $\mathbb{F}_o[[t]]$ be the ideal generated by t in the ring of formal power series $\mathbb{F}[[t]]$. Then $\mathbb{F}_o[[t]] \cong \varprojlim C_i$ as algebras via the isomorphism $t \mapsto (a, a, \dots)$. This inverse limit is very important, as it is one of the central building blocks in the description of the inverse limits of infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$.

To conveniently talk about the inverse limits that arise in the case of infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$, we need a few more definitions. The first two are related to nilpotency.

Definition 3.4

Let A be an associative \mathbb{F} -algebra.

- A is called **residually nilpotent** if $\bigcap_{i \in \mathbb{N}} A^i = \{0\}$.
- A is called **pro-nilpotent** if it is isomorphic to the inverse limit of an infinite sequence of finite-dimensional nilpotent associative \mathbb{F} -algebras.

Remark 3.5

Note that if an associative \mathbb{F} -algebra A is pro-nilpotent, then A is also residually nilpotent.

So far we have only defined coclass for finite-dimensional nilpotent associative \mathbb{F} -algebras. Now we also want to have a notion of coclass for the infinite-dimensional case.

Definition 3.6

We say that an infinite-dimensional associative \mathbb{F} -algebra A has finite coclass r if it is residually nilpotent and for the ascending sequence $cc(A/A^i), i \geq 2$ we have

$$\lim_{i \rightarrow \infty} cc(A/A^i) = r.$$

As in the finite-dimensional case we write $cc(A) = r$.

Lemma 3.7

An infinite-dimensional associative \mathbb{F} -algebra A has finite coclass r if it is residually nilpotent and

$$\sum_{i=1}^{\infty} (\dim(A^i/A^{i+1}) - 1) = r$$

Proof:

This follows because $cc(A/A^j) = \dim(A/A^j) - (j-1) = \sum_{i=1}^{j-1} (\dim(A^i/A^{i+1}) - 1)$ holds for all $j \in \mathbb{N}$. \square

Now that we have the necessary terminology in place, we can give a first elementary lemma from our paper [EM15b] concerning the inverse limits associated to infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$.

Lemma 3.8

Let A be the inverse limit of an infinite path $A_1 \rightarrow A_2 \rightarrow \dots$ in $\mathcal{G}_{\mathbb{F}}(r)$. Then the following holds:

- (a) *Let $c = cl(A_1)$. Then $A/A^{c+i} \cong A_i$ and $cc(A/A^{c+i}) = r$ for all $i \in \mathbb{N}$.*
- (b) *A is an infinite-dimensional pro-nilpotent \mathbb{F} -algebra of coclass r .*
- (c) *A is finitely generated by at most $r+1$ generators.*

Proof:

- (a) By definition of the infinite path one has $A_{i+1}/A_{i+1}^{cl(A_{i+1})} \cong A_i$ for all $i \in \mathbb{N}$. Now by construction of the inverse limit we get $A/A^{c+i} \cong A_i$ for all $i \in \mathbb{N}$. This in turn implies that $cc(A/A^{c+i}) = cc(A_i) = r$, because $A_i \in \mathcal{G}_{\mathbb{F}}(r)$.
- (b) Again by definition of the infinite path in $\mathcal{G}_{\mathbb{F}}(r)$ it follows that $\dim(A_{i+1}) = \dim(A_i) + 1$ for all $i \in \mathbb{N}$. By the first part in (a) $A/A^{c+i} \cong A_i$ holds. Therefore $\dim(A/A^i)$ is unbounded and it follows that A is infinite-dimensional. By construction A is pro-nilpotent and it follows directly from the second part of (a) that it has coclass r .
- (c) Part (b) implies that A is pro-nilpotent and has coclass r . Therefore it follows that $\dim(A/A^2) \leq r+1$ and hence A is finitely generated by at most $r+1$ elements. \square

Recall that two infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$ are called equivalent if one is contained in the other. The next theorem, which is also contained in our paper [EM15b], describes a one-to-one correspondence between infinite-dimensional pro-nilpotent \mathbb{F} -algebras of coclass r and the equivalence classes of infinite paths.

Theorem 3.9

- (a) *Let $A_1 \rightarrow A_2 \rightarrow \dots$ be an infinite path in $\mathcal{G}_{\mathbb{F}}(r)$. Then $\varprojlim A_i$ is an infinite-dimensional pro-nilpotent \mathbb{F} -algebra of coclass r .*
- (b) *Let A be an infinite-dimensional pro-nilpotent \mathbb{F} -algebra of coclass r . Then there is $\ell \in \mathbb{N}$, such that $A/A^\ell \rightarrow A/A^{\ell+1} \rightarrow \dots$ is an infinite path in $\mathcal{G}_{\mathbb{F}}(r)$.*
- (c) *The constructions (a) and (b) yield a one-to-one correspondence between the equivalence classes of infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$ and the isomorphism types of infinite-dimensional pro-nilpotent algebras of coclass r .*

Proof:

- (a) This is Lemma 3.8.
- (b) This follows from the definition of coclass for such infinite-dimensional pro-nilpotent \mathbb{F} -algebras.

- (c) By construction (a) every infinite path in $\mathcal{G}_{\mathbb{F}}(r)$ yields an infinite-dimensional pro-nilpotent \mathbb{F} -algebra of coclass r . Under this construction equivalent paths yield isomorphic algebras and non-equivalent paths yield non-isomorphic algebras. By part (b) every infinite-dimensional pro-nilpotent \mathbb{F} algebra of coclass r is obtained in this way. Therefore the claimed one-to-one correspondence is obtained. \square

The above theorem translates the problem of classifying the infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$ to the problem of classifying the infinite-dimensional pro-nilpotent \mathbb{F} -algebras of coclass r . If we denote by $\mathcal{I}_{\mathbb{F}}(r)$ a set of isomorphism type representatives of infinite-dimensional pro-nilpotent \mathbb{F} -algebras of coclass r , we can formulate the following corollary.

Corollary 3.10

$\mathcal{I}_{\mathbb{F}}(r)$ is a set of isomorphism type representatives of inverse limits of infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$. Thus its elements correspond one-to-one to equivalence classes of infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$.

Our task now is to give a precise structure description for the inverse limits of infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$ and thus provide an analogue of Theorem C from the p -group case for nilpotent associative \mathbb{F} -algebras. Recall from Section 1 that Theorem C in the p -group case states that every inverse limit of an infinite path in a coclass graph of finite p -groups, i.e. every infinite pro- p -group is soluble. In fact one can even describe the structure more precisely as being uniserial p -adic pre-space groups (see Leedham-Green & McKay [LGM02]). Our description for the inverse limits of infinite paths involves the notion of split extensions of algebras, which we recall here.

Definition 3.11

An associative \mathbb{F} -algebra A is said to be a split extension $U \ltimes I$ if U is a subalgebra of A , I is a two-sided ideal in A and $A = \{u + v \mid u \in U, v \in I\}$ and $U \cap I = \{0\}$.

Another ingredient required in the description of the inverse limits of the infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$ is the upper annihilator series of an algebra. In general annihilators play a central role in the theory. Elementary results on annihilators can be found in Chapter I of Kruse & Price [KP69].

Definition 3.12

*Let A be an algebra and let S be a non-empty subset of A . We define the **left, right and two-sided annihilators** of S in A as*

$$\begin{aligned} \text{Ann}_L(S) &= \{a \in A \mid ab = 0 \text{ for all } b \in S\} \\ \text{Ann}_R(S) &= \{a \in A \mid ba = 0 \text{ for all } b \in S\} \\ \text{Ann}(S) &= \{a \in A \mid ab = 0 = ba \text{ for all } b \in S\}. \end{aligned}$$

*We refer to the two-sided annihilator simply as **annihilator**.*

Definition 3.13

We define the upper annihilator series inductively by setting $\text{Ann}_0(A) = \{0\}$ and

$$\text{Ann}_i(A)/\text{Ann}_{i-1}(A) = \text{Ann}(A/\text{Ann}_{i-1}(A)), i \geq 1.$$

Finally, we denote the direct limit of this ascending series by $\text{Ann}_*(A) = \bigcup_{i \in \mathbb{N}} \text{Ann}_i(A)$.

Recall from Example 3.3 that $\mathbb{F}_\circ[[t]]$ denotes the ideal generated by t in $\mathbb{F}[[t]]$ and recall our description (see also [EM15b, Theorem 1]) of the inverse limits of infinite paths in $\mathcal{G}_\mathbb{F}(r)$.

Theorem 1.4

Let \mathbb{F} be an arbitrary field and let r be a non-negative integer. An associative \mathbb{F} -algebra A is isomorphic to an inverse limit of an infinite path in $\mathcal{G}_\mathbb{F}(r)$ if and only if it is a split extension $A = T \ltimes \text{Ann}_*(A)$ with $\dim(\text{Ann}_*(A)) = r$ and T is a subalgebra of A such that $T \cong \mathbb{F}_\circ[[t]]$.

Before we prove Theorem 1.4 we mention the following. We call an infinite-dimensional associative \mathbb{F} -algebra A **just-infinite** if every non-trivial ideal I of A has finite codimension. For a recent paper on properties of just-infinite algebras see Farina & Pendergrass-Rice [FPR07]. The following is a direct corollary of Theorem 1.4.

Corollary 3.14

An inverse limit of an infinite path in $\mathcal{G}_\mathbb{F}(r)$ is just-infinite if and only if it is isomorphic to $\mathbb{F}_\circ[[t]]$ and $r = 0$.

Remark 3.15

The just-infinite groups arising as inverse limits of infinite paths in coclass graphs of finite p -groups are uniserial p -adic space groups and these exist for every coclass.

We now continue with the proof of Theorem 1.4 following our work in [EM15b]. We break it up into several smaller theorems.

Theorem 3.16

Let A be an associative \mathbb{F} -algebra so that $A \cong T \ltimes \text{Ann}_*(A)$ with $\dim(\text{Ann}_*(A)) < \infty$ and again $T \cong \mathbb{F}_\circ[[t]]$. Then A is an infinite-dimensional pro-nilpotent \mathbb{F} -algebra of coclass $\dim(\text{Ann}_*(A))$.

Proof:

Let be $r = \dim(\text{Ann}_*(A))$ and let $t \in A$ be an element that generates T . As T is infinite-dimensional it follows that A is also an infinite-dimensional \mathbb{F} -algebra. Thus it remains to show that (1) A has coclass r and (2) A is pro-nilpotent.

- (1) Because $\text{Ann}_*(A)$ is finite-dimensional, there is $s \in \mathbb{N}$ such that the upper annihilator series of A is given by

$$\{0\} = \text{Ann}_0(A) < \dots < \text{Ann}_s(A) = \text{Ann}_*(A).$$

Now consider $B = A/\text{Ann}_{s-1}(A) \cong T \ltimes \text{Ann}_s(A)/\text{Ann}_{s-1}(A)$. As, by construction of the annihilators, T acts trivially by multiplication on $\text{Ann}_s(A)/\text{Ann}_{s-1}(A)$ one even has $B \cong T \oplus \text{Ann}_s(A)/\text{Ann}_{s-1}(A)$. Thus $B^2 = T^2$ and it follows that $A^2 \leq T^2 \ltimes \text{Ann}_{s-1}(A)$. Iterating this we get $A^{s+1} \leq T^{s+1} \ltimes \text{Ann}_0(A) = T^{s+1}$. Thus the fact that T is residually nilpotent implies that A is residually nilpotent as well. Furthermore, for every $i \geq s$ it follows that

$$\begin{aligned} cc(A/A^{i+1}) &= \dim(A/A^{i+1}) - cl(A/A^{i+1}) \\ &= (\dim(T/T^{i+1}) + \dim(\text{Ann}_*(A))) - i \\ &= i + r - i = r. \end{aligned}$$

This precisely means that A has coclass r .

- (2) We show that A is isomorphic to the inverse limit of the infinite path $A/A^{s+1} \rightarrow A/A^{s+2} \rightarrow \dots$ in $\mathcal{G}_{\mathbb{F}}(r)$. For $i \geq s+1$ consider the natural epimorphisms $\varphi_i : A/A^{i+1} \rightarrow A/A^i$ and let $\pi_i : A \rightarrow A/A^i$ denote the natural projections. Now define

$$\epsilon : A \rightarrow \varprojlim A/A^i, \quad a \mapsto (\pi_{s+1}(a), \pi_{s+2}(a), \dots).$$

This yields an injective \mathbb{F} -algebra homomorphism. To show that ϵ is also surjective note that $A/A^{i+1} \cong T/T^{i+1} \rtimes \text{Ann}_*(A)$ by (1). As this is compatible with the π_i it follows that $\varprojlim A/A^i = \varprojlim T/T^i \rtimes \text{Ann}_*(A)$. Now $\varprojlim T/T^i \cong T$ as seen in Example 3.3. This concludes the proof. \square

The next theorem establishes the existence of a subalgebra $T \cong \mathbb{F}_o[[t]]$ for an inverse limit of an infinite path in $\mathcal{G}_{\mathbb{F}}(r)$.

Theorem 3.17

Let $A \in \mathcal{I}_{\mathbb{F}}(r)$. Then there is an element $t \in A$ such that $t^i \in A^i \setminus A^{i+1}$ for all $i \in \mathbb{N}$. Furthermore, for the subalgebra T generated by t we have $T \cong \mathbb{F}_o[[t]]$.

Proof:

We proceed in three steps. These calculations are inspired by Theorem 1.3.3 in Kruse & Price [KP69].

- (1) Because A has finite coclass, there is $n \in \mathbb{N}$ such that $\dim(A^i/A^{i+1}) = 1$ for all $i \geq n$. We show that there is $t \in A$, so that $t^n + A^{n+1}$ generates A^n/A^{n+1} as \mathbb{F} -vector space. As $A^n = AA^{n-1}$ there is $t \in A$ and $w \in A^{n-1}$ such that $tw \in A^n \setminus A^{n+1}$. As A^n/A^{n+1} is 1-dimensional we can deduce that it is generated as an \mathbb{F} -vector space by $tw + A^{n+1}$. Thus as a set $A^n = \mathbb{F}tw + A^{n+1}$. Let $i \in \{1, \dots, n\}$ be maximal with $A^n = \mathbb{F}t^i w' + A^{n+1}$. Here $w' = w'_1 \cdots w'_{n-i} \in A^{n-i}$. We show that $i = n$ and w' is empty. Assume that $i \neq n$ and so w' is not empty. Then

$$\begin{aligned} A^{n+1} &= A^n A \\ &= (\mathbb{F}t^i w' + A^{n+1})A \\ &= \mathbb{F}t^i w' A + A^{n+2} \\ &= t(t^{i-1} w' A) + A^{n+2} \\ &\subseteq tA^n + A^{n+2} \\ &= t(\mathbb{F}t^i w' + A^{n+1}) + A^{n+2} \\ &\subseteq \mathbb{F}t^{i+1} w' + A^{n+2} \end{aligned}$$

The other inclusion is obvious, hence $\mathbb{F}t^{i+1} w' + A^{n+2} = A^{n+1}$. Note that $t^{i+1} w' \notin A^{n+2}$ as otherwise $A^{n+1} = A^{n+2}$ would follow. Let w'' be such that $w' = w'' w'_{n-i}$. Then $t^{i+1} w'' \in A^n \setminus A^{n+1}$. As A^n/A^{n+1} is 1-dimensional we conclude that $A^n = \mathbb{F}t^{i+1} w'' + A^{n+1}$, which is a contradiction to the choice of i . Thus $i = n$ and $t^n + A^{n+1}$ generates A^n/A^{n+1} as an \mathbb{F} -vector space.

- (2) In this second part we prove that $t^i \in A^i \setminus A^{i+1}$ for all $i \in \mathbb{N}$. By definition $t^i \in A^i$ for all $i \in \mathbb{N}$, so it remains to show that $t^i \notin A^{i+1}$. If $i < n$ and $t^i \in A^{i+1}$, then $t^n = t^{n-i}t^i \in A^{n-i}A^{i+1} = A^{n+1}$ in contradiction to (1). We now proceed by induction to show that $t^i \notin A^{i+1}$ for the remaining $i > n$. So let us suppose that $t^i \notin A^i$ for some $i \in \mathbb{N}$. Then as a set we have $A^i = \mathbb{F}t^i + A^{i+1}$ and therefore $A^{i+1} \subseteq \mathbb{F}t^{i+1} + A^{i+2}$ follows by the same calculation used in Step (1) with w' now being the empty word. Again, the other inclusion is obvious and it follows that $A^{i+1} = \mathbb{F}t^{i+1} + A^{i+2}$. Thus $t^{i+1} \in A^{i+1} \setminus A^{i+2}$ as otherwise $A^{i+1} = A^{i+2}$ would follow. This finishes the induction step.
- (3) Let T be the subalgebra of A generated by the element t constructed in the first step. The second step implies that $\{t^i \mid i \in \mathbb{N}\}$ is an \mathbb{F} -basis for T . In conclusion $T \cong \mathbb{F}_\circ[[t]]$. \square

Finally we prove that the direct limit of the upper annihilator series has the correct dimension.

Theorem 3.18

Let be $A \in \mathcal{I}_\mathbb{F}(r)$. Then $\dim(\text{Ann}_*(A)) = r$.

Proof:

Let $t \in A$ be as in Theorem 3.17 and again let T be the subalgebra generated by t . Recall that the coclass of A is given by

$$r = \text{cc}(A) = \sum_{i=1}^{\infty} (\dim(A^i/A^{i+1}) - 1).$$

Now if $r = 0$, then $\dim(A^i/A^{i+1}) = 1$ for all $i \in \mathbb{N}$ and thus $A = T$ and $\dim(\text{Ann}_*(A)) = 0$ as desired. So let us now suppose that $r > 0$. We proceed in two steps.

- (1) We show that $\text{Ann}(A) \neq \{0\}$.

To this end we explicitly construct an element in $\text{Ann}(A)$. Because $r > 0$ there exists $n \in \mathbb{N}$ with $\dim(A^n/A^{n+1}) \neq 1$ and $\dim(A^i/A^{i+1}) = 1$ for all $i > n$. In this case the set $\{t^i \mid i > n\}$ is a basis for A^{n+1} . Furthermore, there is an element $a + A^{n+1} \in A^n/A^{n+1}$ that is linearly independent of $t^n + A^{n+1}$. Now consider the product ta . We have $ta \in A^{n+1}$, which hence can be written as a sum of t -powers as follows.

$$ta = \sum_{i=n+1}^{\infty} \alpha_i t^i, \quad \alpha_i \in \mathbb{F}$$

and define

$$\bar{a} = a - \sum_{i=n+1}^{\infty} \alpha_i t^{i-1}.$$

Since $a + A^{n+1}$ and $t^n + A^{n+1}$ are linearly independent it follows that $\bar{a} \in A^n \setminus A^{n+1}$ and in particular $\bar{a} \neq 0$. We show that \bar{a} is an element of $\text{Ann}(A)$. By construction

$$t\bar{a} = t \left(a - \sum_{i=n+1}^{\infty} \alpha_i t^{i-1} \right) = ta - \sum_{i=n+1}^{\infty} \alpha_i t^i = 0.$$

Let $b \in A$ be an arbitrary element. Then $\bar{a}b \in A^{n+1}$. Again, write

$$\bar{a}b = \sum_{i=n+1}^{\infty} \beta_i t^i, \quad \beta_i \in \mathbb{F}.$$

It follows that

$$0 = (t\bar{a})b = t(\bar{a}b) = \sum_{i=n+1}^{\infty} \beta_i t^{i+1}$$

and thus $\beta_i = 0$ for all $i > n$. This means that $\bar{a}b = 0$ for all $b \in A$, in particular $\bar{a}t = 0$. Using this fact one can deduce $b\bar{a} = 0$ for all $b \in A$ by a similar calculation. Hence we constructed an element $0 \neq \bar{a} \in \text{Ann}(A)$.

- (2) We iteratively use (1) to show that $\dim(\text{Ann}_*(A)) = r$.

Because $cc(A) = r$ it follows that $\dim_{\mathbb{F}}(A/T) = r$. Obviously $\text{Ann}(T) = \{0\}$, hence $\text{Ann}(A) \cap T = \{0\}$ and this implies $\dim(\text{Ann}(A)) \leq r$. Now consider $B = A/\text{Ann}(A)$ and $S = (T + \text{Ann}(A))/\text{Ann}(A) \leq B$. In this case $S \cong T$ and $cc(B) = \dim_{\mathbb{F}}(B/S) = r - \dim(\text{Ann}(A))$. If $cc(B) = 0$, then $B = S$, $\text{Ann}_*(A) = \text{Ann}(A)$ has dimension r and we are finished. Otherwise we iterate this process with B and S instead of A and T . This terminates after finitely many steps and gives the desired result. \square

We can now combine the previous theorems we proved to give a proof of our structure description for the inverse limits of infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$, i.e. Theorem 1.4.

Proof (of Theorem 1.4):

By Corollary 3.10 A is isomorphic to the inverse limit of an infinite path in $\mathcal{G}_{\mathbb{F}}(r)$ if and only if A is isomorphic to an algebra in $\mathcal{I}_{\mathbb{F}}(r)$.

“ \Rightarrow ”: Suppose that A is isomorphic to an algebra in $\mathcal{I}_{\mathbb{F}}(r)$. Then by Theorem 3.18 we have $\dim(\text{Ann}_*(A)) = r$. In the proof of this theorem we also constructed a subalgebra T isomorphic to $\mathbb{F}_o[[t]]$ such that A is a split extension $T \ltimes \text{Ann}_*(A)$.

“ \Leftarrow ”: Suppose that $A \cong T \ltimes \text{Ann}_*(A)$ with $T \cong \mathbb{F}_o[[t]]$ and $\dim(\text{Ann}_*(A)) = r < \infty$. Then Theorem 3.16 implies that A is isomorphic to an algebra in $\mathcal{I}_{\mathbb{F}}(r)$ \square

Note that the proof of Theorem 3.17 also implies the following corollary. Intuitively it states that all the complexity of a pro-nilpotent \mathbb{F} -algebra of coclass r lies in the first quotients of its power series.

Corollary 3.19

Let A be a pro-nilpotent \mathbb{F} -algebra of coclass r . If $\dim(A^i/A^{i+1}) = 1$ for some $i \in \mathbb{N}$, then $\dim(A^j/A^{j+1}) = 1$ for all $j \geq i$. In particular we have that $\dim(A^j/A^{j+1}) = 1$ for all $j \geq r+1$.

3.2 Construction of infinite paths

This section considers the explicit construction of infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$, i.e. the construction of pro-nilpotent \mathbb{F} -algebras of coclass r . This is done using so called annihilator extensions and cohomological methods.

We begin by recalling basics of extension theory for algebras. Parts of this can be found e.g. in the book on associative algebras by Pierce [Pie82]. Note that we restrict to the case of trivial (bi)modules, as this is sufficient for our aims. If we write A -module, we mean A -bimodule. In this section, unless stated otherwise, let \mathbb{F} denote an arbitrary field, let A be an associative \mathbb{F} -algebra and let I be a finite-dimensional \mathbb{F} -vector space considered as trivial A -module via $I^2 = AI = IA = \{0\}$.

Definition 3.20

An **extension** of A by I is an associative \mathbb{F} -algebra satisfying a short exact sequence

$$0 \rightarrow I \rightarrow E \rightarrow A \rightarrow 0$$

in the category of algebras. We usually identify I with its image in E .

Definition 3.21

We say that an extension E is an **annihilator extension** if $I = \text{Ann}(E)$.

Remark 3.22

Note that from our structure description of the inverse limits of infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$, i.e. Theorem 1.4, we see that these infinite-dimensional pro-nilpotent algebras of coclass r are iterated annihilator extensions of $T \cong \mathbb{F}_o[[t]]$.

Next, we introduce notions to describe the relationship of two extensions.

Definition 3.23

Let E_1 and E_2 be extensions of A by I . We call these extensions

- **isomorphic** if there exists an isomorphism $\iota : E_1 \rightarrow E_2$,
- **strongly isomorphic** if there exists an isomorphism $\iota : E_1 \rightarrow E_2$ with $\iota(I) = I$ and
- **equivalent** if there exists a strong isomorphism $\iota : E_1 \rightarrow E_2$ that induces the identity on I and on $E_1/I \cong A \cong E_2/I$.

The following lemma shows that for annihilator extensions isomorphism and strong isomorphism coincide.

Lemma 3.24

Let E_1 and E_2 be extensions of A by I , such that $\text{Ann}(E_1) = I$ and $\text{Ann}(E_2) = I$. Then strong isomorphism is equivalent to isomorphism.

Proof:

By definition every strong isomorphism is an isomorphism. For the converse let $\iota : E_1 \rightarrow E_2$ be an isomorphism and let $a \in \text{Ann}(E_1)$. Then for all $b \in E_2$ we have $\iota(a)b = \iota(at^{-1}(b)) = \iota(0) = 0$ and similarly $b\iota(a) = 0$. A dual computation shows the reverse inclusion. \square

We look at the following to motivate the definition of cohomology afterwards.

Definition 3.25

Let E be an extension of A by I and consider the projection $\pi : E \rightarrow A$ with kernel I . Let τ be an \mathbb{F} -linear map with $\pi \circ \tau = \text{id}_A$. In this case we call τ a **section**. Note that we can write each element of E uniquely as $\tau(a) + b$ with $a \in A$ and $b \in I$.

Definition 3.26

We call $\rho : A \times A \rightarrow I$, $(a_1, a_2) \mapsto \tau(a_1)\tau(a_2) - \tau(a_1a_2)$ the **2-cocycle** defined by the section τ .

Definition 3.27

We call a k -linear map $\rho : \bigoplus_{j=1}^k A \rightarrow I$ **normalized** if its value is 0 in case one of the input parameters is 0.

Lemma 3.28

The 2-cocycle ρ associated to a section τ is bilinear and normalized.

Proof:

$$\begin{aligned}
\rho(\alpha a_1 + a_2, \beta b_1 + b_2) &= \tau(\alpha a_1 + a_2)\tau(\beta b_1 + b_2) - \tau((\alpha a_1 + a_2)(\beta b_1 + b_2)) \\
&= (\alpha\tau(a_1) + \tau(a_2))(\beta\tau(b_1) + \tau(b_2)) \\
&\quad - \alpha\beta\tau(a_1b_1) - \alpha\tau(a_1b_2) - \beta\tau(a_2b_1) - \tau(a_2b_2) \\
&= \alpha\beta\tau(a_1)\tau(b_1) + \alpha\tau(a_1)\tau(b_2) + \beta\tau(a_2)\tau(b_1) + \tau(a_2)\tau(b_2) \\
&\quad - \alpha\beta\tau(a_1b_1) - \alpha\tau(a_1b_2) - \beta\tau(a_2b_1) - \tau(a_2b_2) \\
&= \alpha\beta\rho(a_1, b_1) + \alpha\rho(a_1, b_2) + \beta\rho(a_2, b_1) + \rho(a_2, b_2)
\end{aligned}$$

for all $\alpha, \beta \in \mathbb{F}$ and a_1, a_2, b_1, b_2 in A . Hence ρ is bilinear. Clearly it is normalized. \square

Remark 3.29

Using the above definitions one can write the multiplication in E as

$$(\tau(a_1) + b_1)(\tau(a_2) + b_2) = \tau(a_1)\tau(a_2) + \tau(a_1a_2) + \rho(a_1, a_2).$$

Evaluating the associativity of E yields

$$\begin{aligned}
((\tau(a_1) + b_1)(\tau(a_2) + b_2))(\tau(a_3) + b_3) &= (\tau(a_1a_2) + \rho(a_1, a_2))(\tau(a_3) + b_3) \\
&= \tau(a_1a_2a_3) + \rho(a_1a_2, a_3), \text{ and} \\
(\tau(a_1) + b_1)((\tau(a_2) + b_2)(\tau(a_3) + b_3)) &= (\tau(a_1) + b_1)(\tau(a_2a_3) + \rho(a_2, a_3)) \\
&= \tau(a_1a_2a_3) + \rho(a_1, a_2a_3)
\end{aligned}$$

for all $a_1, a_2, a_3 \in A$. Thus we have the relation $\rho(a, bc) = \rho(ab, c)$ for all $a, b, c \in A$.

This leads us to the following notions from cohomology theory. We restrict our exposition to the low-dimensional cases. The definition of higher cohomology spaces can be found in Chapter 11 of Pierce [Pie82].

Definition 3.30

We define the following \mathbb{F} -vector spaces.

$$\begin{aligned} C^k(A, I) &= \{\rho : \bigoplus_{j=1}^k A \rightarrow I \mid \rho \text{ } k\text{-linear and normalized}\} \\ Z^2(A, I) &= \{\rho \in C^2(A, I) \mid \rho(a, bc) = \rho(ab, c) \text{ for all } a, b, c \in A\} \\ B^2(A, I) &= \{\rho \in C^2(A, I) \mid \exists \phi \in C^1(A, I) \text{ such that } \rho(a, b) = \phi(ab) \text{ for all } a, b \in A\} \\ H^2(A, I) &= Z^2(A, I) / B^2(A, I) \end{aligned}$$

We call the elements of these spaces *k-cochains*, *2-cocycles*, *2-coboundaries* and *2-cohomology classes*, respectively.

Remark 3.31

We have seen that if τ is a section for an extension of A by I and ρ is the 2-cocycle defined by τ , then $\rho \in Z^2(A, I)$. Conversely every $\rho \in Z^2(A, I)$ arises as a 2-cocycle defined by a section. This implies that $Z^2(A, I)$ corresponds to all possible extensions of A by I .

By Remark 3.31 the space $Z^2(A, I)$ fully describes all extensions of A by I . It remains to solve the isomorphism problem for those extensions. The following Theorem 3.33 solves the problems of classifying extensions up to equivalence and up to strong isomorphism. As we have shown in Lemma 3.24 the latter solves the isomorphism problem in the case of annihilator extensions. For the strong isomorphism case we need the following preliminary lemma.

Lemma 3.32

We can define an action of $\text{Aut}(A) \times \text{Aut}(I)$ on $Z^2(A, I)$ as follows. Let $(\alpha, \beta) \in \text{Aut}(A) \times \text{Aut}(I)$, $\rho \in Z^2(A, I)$ and $a_1, a_2 \in A$. Then (α, β) acts on ρ by

$$\rho^{(\alpha, \beta)}(a_1, a_2) = \beta^{-1}(\rho(\alpha(a_1), \alpha(a_2))).$$

Furthermore, this action leaves $B^2(A, I)$ setwise invariant and hence induces an action on $H^2(A, I)$.

Proof:

The proof that this defines an action of $\text{Aut}(A) \times \text{Aut}(I)$ on $Z^2(A, I)$ is a straightforward calculation. To see that it leaves $B^2(A, I)$ setwise invariant let $\rho \in B^2(A, I)$, i.e. there exists $\phi \in C^1(A, I)$ with $\rho(a_1, a_2) = \phi(a_1 a_2)$ for all $a_1, a_2 \in A$. Let $(\alpha, \beta) \in \text{Aut}(A) \times \text{Aut}(I)$. Then $\rho^{(\alpha, \beta)}(a_1, a_2) = \eta(a_1 a_2)$ for all $a_1, a_2 \in A$, where $\eta = (\beta^{-1} \circ \phi \circ \alpha) \in C^1(A, I)$. \square

Theorem 3.33

Let A be an extension of A by I .

- (1) There is a one-to-one correspondence between the equivalence classes of extensions of A by I and the elements of $H^2(A, I)$.
- (2) There is a one-to-one correspondence between the strong isomorphism classes of extensions of A by I and the orbits of $\text{Aut}(A) \times \text{Aut}(I)$ on the elements of $H^2(A, I)$, where the action is as given in Lemma 3.32.

Proof:

Let E_1 and E_2 be extensions of A by I with sections τ_1 and τ_2 . Let ρ_1 and ρ_2 denote the 2-cocycles defined by τ_1 and τ_2 .

- (1) Suppose that E_1 and E_2 are equivalent extensions. Let $\iota : E_1 \rightarrow E_2$ be an isomorphism that induces the identity on I and on $E_1/I \cong A \cong E_2/I$. This implies that ι has the form $\tau_1(a) + b \mapsto \tau_2(a) + b + \phi(a)$ for $a \in A$, $b \in I$ and some $\phi \in C^1(A, I)$. We have the following for all $a, b \in A$.

$$\begin{aligned}
\rho_2(a, b) - \rho_1(a, b) &= \rho_2(a, b) - \iota(\rho_1(a, b)) \\
&= \tau_2(a)\tau_2(b) - \tau_2(ab) - \iota(\tau_1(a)\tau_1(b) - \tau_1(ab)) \\
&= \tau_2(a)\tau_2(b) - \tau_2(ab) - \iota(\tau_1(a))\iota(\tau_1(b)) + \iota(\tau_1(ab)) \\
&= \tau_2(a)\tau_2(b) - \tau_2(ab) - (\tau_2(a) + \phi(a))(\tau_2(b) + \phi(b)) + (\tau_2(ab) + \phi(ab)) \\
&= \tau_2(a)\tau_2(b) - \tau_2(ab) - \tau_2(a)\tau_2(b) + \tau_2(ab) + \phi(ab) \\
&= \phi(ab)
\end{aligned}$$

Thus $\rho_2 - \rho_1 \in B^2(A, I)$.

Conversely, suppose that $\rho_2 - \rho_1 \in B^2(A, I)$. Then there exists $\phi \in C^1(A, I)$ with $\rho_2(a, b) - \rho_1(a, b) = \phi(ab)$ for all $a, b \in A$. Define $\iota : E_1 \rightarrow E_2 : \tau_1(a) + b \mapsto \tau_2(a) + b + \phi(a)$. Note that ι is linear because τ and ϕ are linear. It remains to show that ι also respects the multiplication. We have

$$\begin{aligned}
\iota((\tau_1(a_1) + b_1)(\tau_1(a_2) + b_2)) &= \iota(\tau_1(a_1a_2) + \rho_1(a_1, a_2)) \\
&= \tau_2(a_1a_2) + \rho_1(a_1, a_2) + \phi(a_1a_2) \\
&= \tau_2(a_1a_2) + \rho_2(a_1, a_2), \text{ and} \\
\iota(\tau_1(a_1) + b_1)\iota(\tau_1(a_2) + b_2) &= (\tau_2(a_1) + b_1 + \phi(a_1))(\tau_2(a_2) + b_2 + \phi(a_2)) \\
&= \tau_2(a_1)\tau_2(a_2) \\
&= \tau_2(a_1a_2) + \rho_2(a_1, a_2)
\end{aligned}$$

for all $a_1, a_2 \in A$ and $b_1, b_2 \in I$. Thus ι is multiplicative. Finally it is easy to verify that ι is bijective and hence an isomorphism.

- (2) Let $\iota : E_1 \rightarrow E_2$ be a strong isomorphism. Define $\alpha^{-1} = \iota|_{E_2/I} \in \text{Aut}(A)$ and $\beta^{-1} = \iota|_I \in \text{Aut}(I)$. Furthermore, define $\phi \in C^1(A, I)$, $a \mapsto \iota(\tau_2(\alpha(a))) - \tau_1(a)$ and let $\gamma \in B^2(A, I)$ be defined by $\gamma(a_1, a_2) = \phi(a_1a_2)$ for all $a_1, a_2 \in A$. We calculate that

$$\begin{aligned}
\iota(\tau_2(\alpha(a_1))\tau_2(\alpha(a_2))) &= \iota(\tau_2(\alpha(a_1)))\iota(\tau_2(\alpha(a_2))) \\
&= (\tau_1(a_1) + \phi(a_1))(\tau_1(a_2) + \phi(a_2)) \\
&= \tau_1(a_1a_2) + \rho_1(a_1, a_2), \text{ and} \\
\iota(\tau_2(\alpha(a_1))\tau_2(\alpha(a_2))) &= \iota(\tau_2(\alpha(a_1)\alpha(a_2)) + \rho_2(\alpha(a_1), \alpha(a_2))) \\
&= \iota(\tau_2(\alpha(a_1)\alpha(a_2))) + \iota(\rho_2(\alpha(a_1), \alpha(a_2))) \\
&= \tau_1(a_1a_2) + \phi(a_1a_2) + \beta^{-1}(\rho_2(\alpha(a_1), \alpha(a_2))) \\
&= \tau_1(a_1a_2) + \gamma(a_1, a_2) + \beta^{-1}(\rho_2(\alpha(a_1), \alpha(a_2)))
\end{aligned}$$

for all $a_1, a_2 \in A$. Comparing the two yields $\rho_1 = \rho_2^{(\alpha, \beta)} + \gamma$ as desired.

Now suppose there is $(\alpha, \beta) \in \text{Aut}(A) \times \text{Aut}(I)$ with $\rho_1 = \rho_2^{(\alpha, \beta)} + \gamma$ for a 2-coboundary γ . Let $\phi \in C^1(A, I)$ be such that $\gamma(a_1, a_2) = \phi(a_1 a_2)$ for all $a_1, a_2 \in A$. Define $\iota : E_2 \rightarrow E_1$, $\tau_2(\alpha(a)) + b \mapsto \tau_1(a) + \beta^{-1}(b) + \phi(a)$. Then a straightforward calculation shows that ι is a strong isomorphism. □

Theorem 3.33 shows that our solution of the strong isomorphism problem of annihilator extensions does require the automorphism group of A . By Remark 3.22 the pro-nilpotent \mathbb{F} -algebras of coclass r that correspond to the inverse limits in $\mathcal{G}_{\mathbb{F}}(r)$ can be constructed as iterated annihilator extensions of $\mathcal{T} \cong \mathbb{F}_o[[t]]$. For the iteration it is very helpful to describe the automorphism group of such an extension. This is done in the following Theorem, which describes how the automorphism group of an extension E of A by I can be described in terms of the automorphism groups $\text{Aut}(A)$ and $\text{Aut}(I)$.

Theorem 3.34

Let E be an extensions of A by I via a section τ and let ρ be the 2-cocycle defined by τ . Consider the map

$$\Psi : \text{Aut}(E) \rightarrow \text{Aut}(A) \times \text{Aut}(I), \delta \mapsto (\delta|_{E/I}, \delta|_I).$$

Then kernel and image of Ψ can be described as follows.

- (1) $\text{im}(\Psi) = \text{Stab}_{\text{Aut}(A) \times \text{Aut}(I)}(\rho + B^2(A, I))$
- (2) $\ker(\Psi) \cong Z^1(A, I) = \{\phi \in C^1(A, I) \mid \phi(ab) = 0 \text{ for all } a, b \in A\}$.

Proof:

- (1) Suppose that $(\alpha, \beta) \in \text{Stab}_{\text{Aut}(A) \times \text{Aut}(I)}(\rho + B^2(A, I))$. In other words we have $\rho(a_1, a_2) = \beta^{-1}(\rho(\alpha(a_1), \alpha(a_2))) + \gamma(a_1, a_2)$ for $\gamma \in B^2(A, I)$ and all $a_1, a_2 \in A$. As usual let $\phi \in C^1(A, I)$ be such that $\phi(a_1 a_2) = \gamma(a_1, a_2)$ for all $a_1, a_2 \in A$. We define $\delta : E \rightarrow E$, $\tau(a) + b \mapsto \tau(\alpha(a)) + \beta(b - \phi(a))$. It is readily verified that δ is additive and a bijection. We calculate

$$\begin{aligned} \delta(\tau(a_1) + b_1)(\tau(a_2) + b_2) &= \delta(\tau(a_1 a_2) + \rho(a_1, a_2)) \\ &= \tau(\alpha(a_1 a_2)) + \beta(\rho(a_1, a_2) - \phi(a_1 a_2)) \\ &= \tau(\alpha(a_1 a_2)) + \rho(a_1, a_2) \\ &= (\tau(\alpha(a_1)) + \beta(b_1 - \phi(a_1)))(\tau(\alpha(a_2)) + \beta(b_2 - \phi(a_2))) \\ &= \delta(\tau(a_1) + b_1)\delta(\tau(a_2) + b_2) \end{aligned}$$

and thus δ is also multiplicative, i.e. $\delta \in \text{Aut}(E)$. Furthermore, $\Psi(\delta) = (\alpha, \beta)$.

Conversely, suppose that $\delta \in \text{Aut}(E)$ is given. Then $\Psi(\delta) = (\alpha, \beta)$ for some $\alpha \in \text{Aut}(A)$ and $\beta \in \text{Aut}(I)$. Hence δ has the form $\tau(a) + b \mapsto \tau(\alpha(a)) + \beta(b) + \phi(a)$ for some

$\phi \in C^1(A, I)$. Now

$$\begin{aligned}\delta((\tau(a_1) + b_1)(\tau(a_2) + b_2)) &= \delta(\tau(a_1 a_2) + \rho(a_1, a_2)) \\ &= \tau(\alpha(a_1 a_2)) + \beta(\rho(a_1, a_2)) + \phi(a_1 a_2), \text{ and} \\ \delta(\tau(a_1) + b_1)\delta(\tau(a_2) + b_2) &= (\tau(\alpha(a_1)) + \beta(b_1) + \phi(a_1))(\tau(\alpha(a_2)) + \beta(b_2) + \phi(a_2)) \\ &= \tau(\alpha(a_1 a_2)) + \rho(\alpha(a_1), \alpha(a_2))\end{aligned}$$

for all $a_1, a_2 \in A$ and $b_1, b_2 \in I$. Comparing the two yields $\beta(\rho(a_1, a_2)) + \phi(a_1 a_2) = \rho(\alpha(a_1), \alpha(a_2))$. Applying β^{-1} on both sides yields

$$\rho(a_1, a_2) + \beta^{-1}(\phi(a_1 a_2)) = \beta^{-1}(\rho(\alpha(a_1), \alpha(a_2))).$$

If one defines $\eta(a_1, a_2) = \beta^{-1}(\phi(a_1 a_2))$ then $\eta \in B^2(A, I)$ and hence the above equation means that $\rho \equiv \rho^{(\alpha, \beta)} \pmod{B^2(A, I)}$. This implies that $\Psi(\delta) \in \text{Stab}_{\text{Aut}(A) \times \text{Aut}(I)}(\rho + B^2(A, I))$ as desired.

- (2) Let $\delta \in \ker(\Psi)$. Then $\delta : E \rightarrow E$, $\tau(a) + b \mapsto \tau(a) + b + \phi_\delta(a)$ for some $\phi_\delta \in C^1(A, I)$. We calculate

$$\begin{aligned}\delta((\tau(a_1) + b_1)(\tau(a_2) + b_2)) &= \delta(\tau(a_1 a_2) + \rho(a_1, a_2)) \\ &= \tau(a_1 a_2) + \rho(a_1, a_2) + \phi_\delta(a_1 a_2), \text{ and} \\ \delta(\tau(a_1) + b_1)\delta(\tau(a_2) + b_2) &= (\tau(a_1) + b_1 + \phi_\delta(a_1))(\tau(a_2) + b_2 + \phi_\delta(a_2)) \\ &= \tau(a_1 a_2) + \rho(a_1, a_2).\end{aligned}$$

As δ is an automorphism, we can deduce that $\phi_\delta(a_1 a_2) = 0$ for all $a_1, a_2 \in A$, i.e. $\phi_\delta \in Z^1(A, I)$.

Conversely, if $\phi \in Z^1(A, I)$ is given, then $\tau(a) + b \mapsto \tau(a) + b + \phi(a)$ is an automorphism in $\ker(\Psi)$. The map $\delta \mapsto \phi_\delta$ is an isomorphism between $\ker(\Psi)$ and $Z^1(A, I)$.

□

We finish this section by giving a theorems that is of interest for the application of these methods to calculate annihilator extensions. In the next chapter we use these methods to prove results on the number of infinite paths for small coclasses. The result considers the dimension of $H^2(A, I)$, which naturally plays an important role.

Theorem 3.35

Let A be finitely presented as (pro-nilpotent) algebra. Let m be the number of relations in the presentation and let I be finite-dimensional of dimension d , say. Then $\dim(H^2(A, I)) \leq md$. In particular $H^2(A, I)$ is also finite-dimensional.

Proof:

Let $A = \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle$ and let b_1, \dots, b_d denote a basis of I . Then every extension of A by I has a presentation on the generators $\{a_1, \dots, a_n, b_1, \dots, b_d\}$ and relations of the

following form

$$\begin{aligned} a_i b_j &= 0 & \text{for } 1 \leq i \leq n, 1 \leq j \leq d \\ b_j a_i &= 0 & \text{for } 1 \leq i \leq n, 1 \leq j \leq d \\ b_i b_j &= 0 & \text{for } 1 \leq i, j \leq d \\ r_i &= \sum_{j=1}^d \alpha_{i,j} b_j & \text{for } 1 \leq i \leq m \end{aligned}$$

with some $\alpha_{i,j} \in \mathbb{F}$. Hence we obtain a map

$$\gamma : Z^2(A, I) \rightarrow \mathbb{F}^{d \times m}.$$

Note that γ is \mathbb{F} -linear and that its kernel is contained in $B^2(A, I)$. This implies that $H^2(A, I) \cong Z^2(A, I)/B^2(A, I) \cong \gamma(Z^2(A, I))/\gamma(B^2(A, I))$ is isomorphic to a subquotient of $\mathbb{F}^{d \times m}$ and thus finite-dimensional. \square

Remark 3.36

Of course $\gamma(Z^2(A, I))$ and $\gamma(B^2(A, I))$ depend on the chosen presentation, but the isomorphism type of the quotient does not. We will usually calculate $H^2(A, I)$ using the map γ .

3.3 Number of infinite paths

In this chapter we are concerned with the number of (equivalence classes) of infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$, which we denote by $n_{\mathbb{F}}(r) = |\mathcal{I}_{\mathbb{F}}(r)|$. In particular we prove that the analogue of Theorem D from the p -group case (see 1) holds, i.e. the number of equivalence classes of infinite paths is finite, if and only if the field \mathbb{F} is finite or $r \leq 1$. Furthermore, we give an explicit number in the case $r \leq 2$. Note that these results are also contained in our paper [EM15b], but in contrast to the given reference here we use the cohomological methods developed in Section 3.2 to obtain and verify them.

We begin by proving the following lemma for pro-nilpotent \mathbb{F} -algebras of coclass r with $r + 1$ generators. Recall that this is the maximal possible number of generators for such an algebra.

Lemma 3.37

Let \mathbb{F} be an arbitrary field and let $r \in \mathbb{N}_0$. Let $A \in \mathcal{I}_{\mathbb{F}}(r)$ with $\dim(A/A^2) = r + 1$, then $A = T \oplus \text{Ann}(A)$, where $T \cong \mathbb{F}_{\circ}[[t]]$.

Proof:

In this situation A is an annihilator extension of $T \cong \mathbb{F}_{\circ}[[t]]$ by $I \cong \mathbb{F}^r$. As pro-nilpotent algebra T has the presentation $\langle t \rangle$ with one generator and no relations. Therefore Theorem 3.35 implies that $H^2(A, I) = \{0\}$ and the result follows. \square

Lemma 3.37 has the following immediate corollary.

Corollary 3.38

Let \mathbb{F} be an arbitrary field, $r \in \mathbb{N}_0$ and $A \in \mathcal{I}_{\mathbb{F}}(r)$.

- If $r = 0$, then $A \cong \mathbb{F}_o[[t]]$.
- If $r = 1$, then $A \cong \mathbb{F}_o[[t]] \oplus \text{Ann}(A)$ with $\text{Ann}(A)$ of dimension 1.

The next theorem gives an upper bound for $n_{\mathbb{F}}(r)$ in case the field \mathbb{F} is finite. Note that this bound is not sharp.

Theorem 3.39

Let \mathbb{F} be a finite field and let $r \in \mathbb{N}_0$. Then $n_{\mathbb{F}}(r) \leq |\mathbb{F}|^{r^3+2r^2}$. In particular $n_{\mathbb{F}}(r)$ is finite.

Proof:

Let be $A \in \mathcal{I}_{\mathbb{F}}(r)$. By our structure description Theorem 1.4 we know that $A = T \ltimes \text{Ann}_*(A)$ with $\dim(\text{Ann}_*(A)) = r$. Hence A is completely described by the structure constants for $\text{Ann}_*(A)$ and the elements tb_i, b_it for $i \in \{1, \dots, r\}$. Note that tb_i and b_it are elements of $\text{Ann}_*(A)$. It only remains to count the possibilities. There are at most $|\mathbb{F}|^{r^3}$ structure constant tables for $\text{Ann}_*(A)$ and at most $|\mathbb{F}|^{2r^2}$ possibilities to choose the elements tb_i and b_it , thus $n_{\mathbb{F}}(r) \leq |\mathbb{F}|^{r^3+2r^2}$. \square

In the remaining part of this section we consider the case of coclass 2 in more detail.

- (1) The first case are annihilator extensions of $T \cong \mathbb{F}_o[[t]]$ by a 2-dimensional \mathbb{F} -vector space. By Lemma 3.37 there is exactly one isomorphism type of such an extension, namely the direct sum.
- (2) The second case are annihilator extensions of \mathfrak{A} by \mathfrak{J} , where $\mathfrak{A} = \langle t, a \mid at = ta = a^2 = 0 \rangle$ is a representative of the unique isomorphism type of pro-nilpotent \mathbb{F} -algebras of coclass 1 and \mathfrak{J} is a 1-dimensional \mathbb{F} -vector space considered as trivial \mathfrak{A} -module.

To this end we first construct all extensions of \mathfrak{A} by \mathfrak{J} up to strong isomorphism. Let b be a basis element for \mathfrak{J} and let E be an extension of \mathfrak{A} by \mathfrak{J} defined by a 2-cocycle $\rho \in Z^2(\mathfrak{A}, \mathfrak{J})$. Then E has a presentation on the generators t, a, b with relations

$$\begin{aligned} at &= \rho(a, t) = \alpha_1 b \\ ta &= \rho(t, a) = \alpha_2 b \\ a^2 &= \rho(a, a) = \alpha_3 b \\ bt &= tb = ab = ba = b^2 = 0 \end{aligned}$$

and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}$. Thus we obtain an \mathbb{F} -linear $\gamma : Z^2(\mathfrak{A}, \mathfrak{J}) \rightarrow \mathbb{F}^3, \rho \mapsto (\alpha_1, \alpha_2, \alpha_3)^T$ as in Theorem 3.35. The map γ has the following properties.

Lemma 3.40

The map γ is surjective and $\gamma(B^2(\mathfrak{A}, \mathfrak{J})) = \{(0, 0, 0)^T\}$. Hence $H^2(\mathfrak{A}, \mathfrak{J}) \cong Z^2(\mathfrak{A}, \mathfrak{J}) \cong \mathbb{F}^3$.

Proof:

The image of γ contains those elements of \mathbb{F}^3 , such that the \mathbb{F} -algebra defined by these elements is associative. Evaluating all relations yields that all elements of \mathbb{F}^3 give associative \mathbb{F} -algebras, hence γ is surjective. For the second part let $\phi \in C^1(\mathfrak{A}, \mathfrak{J})$ and let $\rho \in B^2(\mathfrak{A}, \mathfrak{J})$ defined by $\rho(a_1, a_2) = \phi(a_1 a_2)$ for all $a_1, a_2 \in \mathfrak{A}$. Then $\rho(a, t) = \phi(at) = \phi(0) = 0$ and similarly $\rho(t, a) = \rho(a, a) = 0$. Therefore $\gamma(\rho) = (0, 0, 0)^T$ and hence $\gamma(B^2(\mathfrak{A}, \mathfrak{J})) = \{(0, 0, 0)^T\}$

as claimed. \square

To classify the extensions of \mathfrak{A} by \mathfrak{J} up to strong isomorphism, by Theorem 3.33 it remains to determine the action of $\text{Aut}(\mathfrak{A}) \times \text{Aut}(\mathfrak{J})$ on $H^2(\mathfrak{A}, \mathfrak{J})$. Clearly $\text{Aut}(\mathfrak{J}) \cong \mathbb{F}^*$. The automorphism group of \mathfrak{A} is determined in the next lemma.

Lemma 3.41

Let $\varphi \in \text{Aut}(\mathfrak{A})$. Then $\varphi(a) = \alpha a$ for some $\alpha \neq 0$ and $\varphi(t) = \sum_{i=1}^{\infty} \beta_i t^i + \delta a$ for some $\beta_i, \delta \in \mathbb{F}$ and $\beta_1 \neq 0$. Note that all possible values of α, β_i and δ yield elements in $\text{Aut}(\mathfrak{A})$.

Proof:

Let T be the subalgebra of \mathfrak{A} generated by t . Write $\varphi(a) = s + \alpha a$ with $s \in T$ and $\alpha \in \mathbb{F}$ and. Now $0 = \varphi(0) = \varphi(a^2) = \varphi(a)^2 = (s + \alpha a)^2 = s^2 + \alpha s a + \alpha a s + \alpha^2 a^2 = s^2$ implies $s = 0$, hence $\varphi(a) = \alpha a$. Under this assumption φ is a homomorphism. Surjectivity of φ is equivalent to $\alpha \neq 0$ and $\beta_1 \neq 0$. \square

This now allows us to determine the actions of $\text{Aut}(\mathfrak{A})$ and $\text{Aut}(\mathfrak{J})$ on $H^2(\mathfrak{A}, \mathfrak{J}) \cong \mathbb{F}^3$.

Lemma 3.42

$\text{Aut}(A)$ acts on $H^2(\mathfrak{A}, \mathfrak{J}) \cong \mathbb{F}^3$ as

$$\left\langle \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix}, \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \delta \\ 0 & 1 & \delta \\ 0 & 0 & 1 \end{pmatrix} \mid \alpha, \beta_1 \in \mathbb{F}^*, \delta \in \mathbb{F} \right\rangle.$$

and $\text{Aut}(I)$ acts on $H^2(\mathfrak{A}, \mathfrak{J}) \cong \mathbb{F}^3$ as

$$\left\langle \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \mid \epsilon \in \mathbb{F}^* \right\rangle.$$

Proof:

This follows directly by evaluating the action of the automorphisms

$$\begin{array}{lll} a \mapsto \alpha a & a \mapsto a & a \mapsto a \\ t \mapsto t & t \mapsto \sum_{i=1}^{\infty} \beta_i t^i t & t \mapsto t + \delta a \end{array}$$

in $\text{Aut}(\mathfrak{A})$ and of $b \mapsto \epsilon b$ in $\text{Aut}(\mathfrak{J})$. \square

The following lemma describes the orbits of $H^2(\mathfrak{A}, \mathfrak{J})$ under this action and hence gives a classification of extensions of \mathfrak{A} by \mathfrak{J} up to strong isomorphism.

Lemma 3.43

There are $|\mathbb{F}| + 4$ orbits on $H^2(\mathfrak{A}, \mathfrak{J})$ under the action of $\text{Aut}(\mathfrak{A}) \times \text{Aut}(\mathfrak{J})$. A complete set of orbit representatives is given by $\{(0, 0, 0)^T, (0, 0, 1)^T, (0, 1, 1)^T, (0, 1, 0)^T, (1, x, 0)^T \mid x \in \mathbb{F}\}$.

Proof:

We begin with the easy cases.

- (i) The orbit of $(0, 0, 0)^T$ only contains this single element.
- (ii) The orbit of $(0, 0, 1)^T$ contains all elements $(0, 0, f)^T$ with $f \in \mathbb{F}^*$.
- (iii) The orbit of $(0, 1, 0)^T$ contains all elements $(0, f, 0)^T$ with $f \in \mathbb{F}^*$.
- (iv) The orbit of $(1, x, 0)^T$ contains all elements $(f, fx, 0)^T$ with $f \in \mathbb{F}^*$ and $x \in \mathbb{F}$.

The above orbits are obviously disjoint. It remains to show that the orbit of $(0, 1, 1)^T$ is disjoint with all the already calculated orbits and that it covers all remaining elements of \mathbb{F}^3 . We only show that the union of the orbits contains all elements of \mathbb{F}^3 , because the disjointness can be easily verified. We calculate the following, where as in Lemma 3.42 we require that $\alpha, \beta_1, \epsilon \in \mathbb{F}^*$ and $\delta \in \mathbb{F}$.

$$\begin{aligned}
 \text{(a)} \quad & \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \epsilon \\ \epsilon \end{pmatrix} \\
 \text{(b)} \quad & \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \epsilon \\ \epsilon \end{pmatrix} = \begin{pmatrix} 0 \\ \beta_1 \epsilon \\ \epsilon \end{pmatrix} \\
 \text{(c)} \quad & \begin{pmatrix} 1 & 0 & \delta \\ 0 & 1 & \delta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \beta_1 \epsilon \\ \epsilon \end{pmatrix} = \begin{pmatrix} \delta \epsilon \\ (\beta_1 + \delta) \epsilon \\ \epsilon \end{pmatrix}
 \end{aligned}$$

Let $f, g \in \mathbb{F}^*$. We see that the following holds.

- $(0, 0, 0)^T$ is covered by (i).
- $(0, 0, f)^T$ is covered by (ii).
- $(0, f, 0)^T$ is covered by (iii).
- $(f, 0, 0)^T$ is covered by (iv) with $x = 0$.
- $(f, g, 0)^T$ is covered by (iv) with $x \neq 0$.
- $(0, f, g)^T$ is covered by (b).
- $(f, 0, g)^T$ is covered by (c) with $\beta_1 = -\delta$.
- All remaining triples are covered by (c) with $\beta_1 \neq -\delta$.

□

We finally observe that E is an annihilator extensions of \mathfrak{A} by \mathfrak{J} if and only $\gamma(\rho) \neq (0, 0, 0)^T$ and thus in conclusion we have proved the following Theorem.

Theorem 3.44

Let \mathbb{F} be an arbitrary field. Then $n_{\mathbb{F}}(2) = |\mathbb{F}| + 4$.

We can combine the results of this section to a proof of Theorem 1.5, which we recall here.

Theorem 1.5

The number of equivalence classes of infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$ is finite if and only if $r \leq 1$ or \mathbb{F} is finite.

Proof:

By Corollary 3.38 and Theorem 3.39 the number of equivalence classes of infinite paths is finite if \mathbb{F} is finite or $r \leq 1$. Theorem 3.44 shows that for an infinite field \mathbb{F} there are infinitely many equivalence classes of infinite paths in $\mathcal{G}_{\mathbb{F}}(2)$. If A is an algebra in $\mathcal{I}_{\mathbb{F}}(2)$, then for $r \geq 3$ we have that $A \oplus \mathcal{Z}_{r-2} \in \mathcal{I}_{\mathbb{F}}(r)$ and hence for infinite fields \mathbb{F} there are infinitely many equivalence classes of infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$ for all $r \geq 2$. \square

Chapter 4

Trees in coclass graphs

This chapter has overlap with our paper [EM16].

4.1 Roots of maximal descendant trees

In this section we prove results on the broad shape of the graphs $\mathcal{G}_{\mathbb{F}}(r)$. By definition $\mathcal{G}_{\mathbb{F}}(r)$ is a forest and the union of the maximal descendant trees contained in it. We prove that the roots of maximal descendant trees in $\mathcal{G}_{\mathbb{F}}(r)$ have dimension at most $2r$ and hence $\mathcal{G}_{\mathbb{F}}(r)$ is a union of finitely many maximal descendant trees in the case that \mathbb{F} is finite. With the results proven in Section 3.1 we can then prove that $\mathcal{G}_{\mathbb{F}}(r)$ consists of finitely many maximal coclass trees and finitely many other vertices.

We begin by mentioning that an analogue of Corollary 3.19 also holds for finite-dimensional nilpotent associative \mathbb{F} -algebras. This result is also contained in our paper [EM16].

Theorem 4.1

Let A be a nilpotent associative \mathbb{F} -algebra. If $\dim(A^i/A^{i+1}) = 1$ for some $i < \text{cl}(A)$, then $\dim(A^j/A^{j+1}) = 1$ for all $i \leq j \leq \text{cl}(A)$.

Proof:

If $n = \text{cl}(A)$, then there is nothing to show. Hence we assume that $n < \text{cl}(A)$ so that $A^{n+1} \neq A^{n+2}$. We now proceed in two steps.

(1): We show that there exists an element $t \in A$ such that A^n/A^{n+1} is generated by t^n as an \mathbb{F} -vector space. As $\dim(A^n/A^{n+1}) = 1$ and $A^n = AA^{n-1}$, it follows that there exist $t \in A$ and $w \in A^{n-1}$ with $A^n = \mathbb{F}tw + A^{n+1}$. Using the argument of Step (1) in the proof of Theorem 3.17, it follows that $A^n = \mathbb{F}t^n + A^{n+1}$. Thus (1) follows.

(2): We show that $A^i = \mathbb{F}t^i + A^{i+1}$ for each $n \leq i \leq \text{cl}(A)$. First, $t^i \in A^i$ for each $1 \leq i \leq \text{cl}(A)$ by the definition of A^i . Further, $t^i \notin A^{i+1}$ for $1 \leq i \leq n$, as otherwise $t^n = t^{n-i}t^i \in A^{n-i}A^{i+1} = A^{n+1}$ in contradiction to (1). We use induction on i to show the desired result for $n \leq i \leq \text{cl}(A)$. The initial step of this induction is proved in (1). Thus we assume that $A^i = \mathbb{F}t^i + A^{i+1}$ for some $i \geq n$. Then again using the argument of Step (1) in the proof of Theorem 3.17 we deduce that $A^{i+1} \subseteq \mathbb{F}t^{i+1} + A^{i+2}$. As $\mathbb{F}t^{i+1} + A^{i+2} \subseteq A^{i+1}$, it follows that $A^{i+1} = \mathbb{F}t^{i+1} + A^{i+2}$. This completes the induction step and yields (2). \square

This again intuitively means that all the complexity lies in the first quotients of the power series. In other words we have the following theorem.

Theorem 4.2

Let A be a nilpotent associative \mathbb{F} -algebra of coclass r . If $r = 0$ then $cc(A/A^i) = 0$ for all $i \geq 2$ and if $r \geq 1$ we have $cc(A/A^i) = r$ for all $i \geq r + 1$.

Proof:

The theorem obviously holds for $r = 0$. For $r \geq 1$ recall that the coclass of A/A^i is given by

$$cc(A/A^i) = \sum_{j=1}^{i-1} (\dim(A^j/A^{j+1}) - 1).$$

The result now follows from Theorem 4.1. □

This result now enables us to prove Theorem 1.6, i.e. bounds for the dimension of a root of a maximal descendant tree of $\mathcal{G}_{\mathbb{F}}(r)$. The upper bound (without sharpness) is also proved in our paper [EM16].

Theorem 1.6

Let \mathbb{F} be an arbitrary field and let r be a non-negative integer. Let R be a root of a maximal descendant tree in $\mathcal{G}_{\mathbb{F}}(r)$. If $r = 0$, then $\dim(R) = 1$. If $r \geq 1$, then $r + 1 \leq \dim(R) \leq 2r$ and these bounds are sharp.

Proof:

The case $r = 0$ is trivial, hence we assume that $r \geq 1$. For the lower bound note that $\dim(R) = cl(R) + cc(R) \geq 1 + r$. Furthermore, the algebra \mathcal{Z}_{r+1} from Example 2.5 has coclass r and dimension $r + 1$, hence this lower bound is sharp.

For the upper bound let A be an algebra in $\mathcal{G}_{\mathbb{F}}(r)$ and let $B = A/A^{r+1}$ be its class r quotient. Theorem 4.2 implies that $cc(B) = r$. Hence $\dim(B) = cl(B) + cc(B) = 2r$. By construction $A \in \mathcal{T}(B)$ and $\mathcal{T}(B)$ in turn is contained in a maximal descendant tree of $\mathcal{G}_{\mathbb{F}}(r)$. If R is the root of this maximal descendant tree, then $\dim(R) \leq 2r$ follows. For the sharpness of the upper bound consider the algebra $\langle a \mid a^{r+1} \rangle \oplus \langle b \mid b^{r+1} \rangle$. This algebra has coclass r , dimension $2r$ and all quotients have coclass smaller than r . Hence this algebra is a root of a maximal descendant tree of $\mathcal{G}_{\mathbb{F}}(r)$. □

Theorems 4.1 and Theorem 1.6 also enable us to prove a class bound for the coclass settledness of a nilpotent associative \mathbb{F} -algebra.

Definition 4.3

*We call a finite-dimensional nilpotent associative \mathbb{F} -algebra A **coclass settled** if all its immediate descendants have the same coclass as A .*

Corollary 4.4

Let A be a finite-dimensional nilpotent associative \mathbb{F} -algebra of coclass r . If $cl(A) \geq r + 1$, then A is coclass settled.

Proof:

If A is terminal or $r = 0$, then the theorem holds trivially. Suppose that A is capable and $r \geq 1$. By assumption $cl(A) \geq r + 1$, i.e. $dim(A) \geq 2r + 1$ and hence by Theorem 1.6 A is not a root in $\mathcal{G}_{\mathbb{F}}(r)$. Hence it is an immediate descendant of another algebra in $\mathcal{G}_{\mathbb{F}}(r)$ and it follows that $dim(A^{cl(A)}/A^{cl(A)+1}) = 1$. Let B be an immediate descendant of A . Then $B^{cl(B)-1}/B^{cl(B)} \cong A^{cl(A)}/A^{cl(A)+1}$ and hence $dim(B^{cl(B)-1}/B^{cl(B)}) = 1$. Therefore, by Theorem 4.1 also $dim(B^{cl(B)}/B^{cl(B)+1}) = 1$ which implies $cc(B) = cc(A)$. \square

Remark 4.5

In the case of finite p -groups Shalev shows in [Sha94, Corollary 4.4] that 2-groups are coclass settled by class 2^{r+3} and p -groups for odd p are coclass settled by class $2p^r$.

Furthermore, Theorem 1.6 also has the following immediate corollary describing the broad shape of $\mathcal{G}_{\mathbb{F}}(r)$ for finite fields.

Corollary 1.7

Let \mathbb{F} be an arbitrary finite field and let r be a non-negative integer. Then $\mathcal{G}_{\mathbb{F}}(r)$ is a disjoint union of finitely many maximal descendant trees.

We can further strengthen this result using the following theorem from our paper [EM16], which considers branching of infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$.

Theorem 4.6

Let \mathbb{F} be an arbitrary finite field and let r be a non-negative integer. Let A be an algebra of class at least $2r$ in $\mathcal{G}_{\mathbb{F}}(r)$. Then its descendant tree $\mathcal{T}(A)$ contains at most one infinite path starting at its root; that is, either $\mathcal{T}(A)$ has no infinite path or it is a coclass tree.

Proof:

The result holds for $r \leq 1$, because $\mathcal{G}_{\mathbb{F}}(r)$ contains only one maximal infinite path in these cases by Corollary 3.38. Hence in the following we assume that $r \geq 2$. Let c denote the class of A and let $\mathcal{T}(A)$ have an infinite path $A = A_1 \rightarrow A_2 \dots$, say. Let \bar{A} be the inverse limit associated to this infinite path. By Theorem 3.18 $Ann_*(\bar{A}) = Ann_r(\bar{A})$ is an r -dimensional ideal in \bar{A} . Additionally, there exists an element $t \in \bar{A}$ with $t^{r+1} \neq 0$ and every such element t generates an infinite-dimensional algebra $T = \langle t \rangle$ satisfying that $\bar{A} = T \ltimes Ann_r(\bar{A})$. This implies that $\bar{A}^j \leq T^j \ltimes Ann_{r-j+1}(\bar{A})$ for each $j \in \mathbb{N}$ and thus $\bar{A}^j = T^j$ for each $j \geq r + 1$.

We identify A with its isomorphic copy \bar{A}/\bar{A}^{c+1} and let $\lambda : \bar{A} \rightarrow A$ denote the associated epimorphism. Let $B = A/A^{r+1}$ with associated epimorphism $\varphi : A \rightarrow B$. and let $\sigma : \bar{A} \rightarrow B$ be the composition of λ and φ . We show that the isomorphism type of \bar{A} is fully defined by A and B . This is sufficient to prove the theorem. If there are two infinite paths in \mathcal{T}_A starting at A , then they both have A and B as quotients and hence have isomorphic inverse limits. Because an inverse limit defines the algebras on its associated infinite path via its power series quotients, we have that the two considered infinite paths are identical, which yields the desired result.

We now construct \bar{A} from A and B . Let $a \in A$ with $a^c \neq 0$ and let $b = \varphi(a) \in B$. The element a exists, as A is a quotient of \bar{A} . Write $M = \varphi(Ann_r(A)) \leq B$ and let $X = \langle x \rangle$

denote the ideal generated by x in the ring of formal power series $\mathbb{F}[[x]]$. We define an action of X on M via $m \cdot x = mb$ and $x \cdot m = bm$ for each $m \in M$. Let

$$\bar{B} = X \ltimes M.$$

Then \bar{B} is defined from A and B only. We show that \bar{A} and \bar{B} are isomorphic. For this we choose $s \in \bar{A}$ so that $\lambda(s) = a$. Then $s^c \neq 0$. As $c \geq 2r \geq r + 1$, it follows that $S = \langle s \rangle$ is an infinite-dimensional subalgebra of \bar{A} with $\bar{A} = S \ltimes \text{Ann}_r(\bar{A})$. Further, $S \rightarrow X : s \mapsto x$ induces an algebra isomorphism. We now write the elements of \bar{A} as $f(s) + w$, where $f(s)$ is a formal power series in S and $w \in \text{Ann}_r(\bar{A})$. Next, we define

$$\gamma : \bar{A} \rightarrow \bar{B} : f(s) + w \mapsto f(x) + \sigma(w).$$

It remains to show that this is an algebra isomorphism.

We observe that $\sigma(\text{Ann}_r(\bar{A})) = \varphi(\lambda(\text{Ann}_r(\bar{A}))) \leq \varphi(\text{Ann}_r(\lambda(\bar{A}))) = \varphi(\text{Ann}_r(A)) = M$ and thus σ maps $\text{Ann}_r(\bar{A})$ into M . Next, $\ker(\sigma) = \bar{A}^{r+1} = S^{r+1}$ and thus $\ker(\sigma) \cap \text{Ann}_r(\bar{A}) = \{0\}$. It follows that σ restricted to $\text{Ann}_r(\bar{A})$ is injective. As $c \geq 2r \geq r + 1$, we deduce that $\bar{A}^{c+1} = S^{c+1}$. Hence $A \cong \bar{A}/\bar{A}^{c+1} \cong S/S^{c+1} \ltimes \text{Ann}_r(\bar{A})$. Using this, we inductively obtain that $\text{Ann}_j(A) \cong \text{Ann}_j(S/S^{c+1} \ltimes \text{Ann}_r(\bar{A})) = S^{c+1-j}/S^{c+1} \ltimes \text{Ann}_j(\bar{A})$. As $c + 1 - r \geq 2r + 1 - r = r + 1$, it follows that $S^{c+1-r}/S^{c+1} \leq \ker(\varphi)$ and thus $\dim(M) = \dim(\varphi(\text{Ann}_r(A))) = \dim(\text{Ann}_r(\bar{A}))$. Hence $\sigma : \text{Ann}_r(\bar{A}) \rightarrow M$ is an algebra isomorphism.

As σ is \mathbb{F} -linear and bijective, we have that γ is \mathbb{F} -linear and bijective. Finally, we need to show that γ respects the multiplication. We note that $\sigma(s) = \varphi(\lambda(s)) = \varphi(a) = b$ and thus $\sigma(f(s)w) = f(\sigma(s))\sigma(w) = f(b)\sigma(w)$ for each formal power series f and each $w \in \text{Ann}_r(\bar{A})$. Further, $f(b)\sigma(w) = f(x)\sigma(w)$ by the definition of the action of X on M . Similarly, $\sigma(wf(s)) = \sigma(w)f(b)$ and $\sigma(w)f(b) = \sigma(w)f(x)$. This yields

$$\begin{aligned} & \gamma(f_1(s) + w_1)\gamma(f_2(s) + w_2) \\ &= (f_1(x) + \sigma(w_1))(f_2(x) + \sigma(w_2)) \\ &= f_1(x)f_2(x) + f_1(x)\sigma(w_2) + \sigma(w_1)f_2(x) + \sigma(w_1)\sigma(w_2) \\ &= f_1(x)f_2(x) + f_1(b)\sigma(w_2) + \sigma(w_1)f_2(b) + \sigma(w_1w_2) \\ &= f_1(x)f_2(x) + \sigma(f_1(s)w_2) + \sigma(w_1f_2(s)) + \sigma(w_1w_2) \\ &= f_1(x)f_2(x) + \sigma(w_1w_2 + f_1(s)w_2 + w_1f_2(s)) \\ &= \gamma(f_1(s)f_2(s) + w_1w_2 + f_1(s)w_2 + w_1f_2(s)) \\ &= \gamma((f_1(s) + w_1)(f_2(s) + w_2)) \end{aligned}$$

for all $f_1(s) + w_1, f_2(s) + w_2 \in \bar{A}$. Hence γ is an algebra isomorphism and we have proved the claim. \square

The broad shape of $\mathcal{G}_{\mathbb{F}}(r)$ for finite fields can now be described by Theorem 1.9 (see also [EM16]).

Theorem 1.9

Let \mathbb{F} be an arbitrary finite field and let r be a non-negative integer. Then each maximal

descendant tree of $\mathcal{G}_{\mathbb{F}}(r)$ consists of finitely many maximal coclass trees and finitely many other vertices. Hence $\mathcal{G}_{\mathbb{F}}(r)$ is a disjoint union of finitely many maximal coclass trees and finitely many other vertices.

Proof:

Let \mathbb{F} be a finite field and r a non-negative integer. Then, since \mathbb{F} is finite, a maximal descendant tree \mathcal{T} in $\mathcal{G}_{\mathbb{F}}(r)$ has only finitely many algebras of class $2r$, i.e. dimension $3r$. Thus by Theorem 4.6 it follows that \mathcal{T} contains only finitely many coclass trees and finitely many other vertices. \square

We give the following example to illustrate the difference between maximal descendant trees and maximal coclass trees.

Example 4.7

Consider the following infinite-dimensional pro-nilpotent \mathbb{F}_2 -algebras of coclass 3:

$$A = \langle t, a, b, c \mid a^2 - c, ab, ac, ba, b^2, bc, ca, cb, c^2, ta - b, tb, tc, at, bt, ct \rangle,$$

$$B = \langle t, a, b, c \mid a^2 - c, ab, ac, ba, b^2, bc, ca, cb, c^2, ta, tb, tc, at - b, bt, ct \rangle.$$

We have that $cc(A/A^3) = cc(B/B^3) = 3$ and $A/A^3 \cong B/B^3$. Hence the infinite paths described by A and B are contained in the same maximal descendant tree. As $A/A^4 \not\cong B/B^4$ we deduce that this maximal descendant tree contains the two maximal coclass trees with roots A/A^4 and B/B^4 .

4.2 Depth of coclass trees

Next we consider the depth of a maximal coclass tree \mathcal{T} of $\mathcal{G}_{\mathbb{F}}(r)$. Again, as $\mathcal{G}_{\mathbb{F}}(0)$ is completely understood, we suppose that $r > 0$. We show that for almost all branches \mathcal{B} of \mathcal{T} we have $dep(\mathcal{B}) \leq r$ and hence \mathcal{T} has bounded depth. To establish this result, we need to work with specific presentations for (almost all) algebras in \mathcal{T} . Note that we refer to the algebras on the infinite path of a coclass tree as **mainline algebras**.

Lemma 4.8

Let \mathcal{T} be a maximal coclass tree in $\mathcal{G}_{\mathbb{F}}(r)$. Then for $c \geq r$ the class c mainline algebra of \mathcal{T} can be described by a presentation of the form

$$\begin{aligned} \langle t, b_1, \dots, b_r \mid & t^{c+1}, \\ & b_i b_j - s_{ij}, \text{ for } 1 \leq i, j \leq r \\ & t b_i - s_{0i}, \text{ for } 1 \leq i \leq r, \\ & b_i t - s_{i0}, \text{ for } 1 \leq i \leq r \rangle \end{aligned}$$

where s_{ij} lies in the subspace $\langle b_1, \dots, b_r \rangle$ for $1 \leq i, j \leq r$, $s_{0i} = \sum_{j=i+1}^r \alpha_j b_j$ and $s_{i0} = \sum_{j=i+1}^r \overline{\alpha_j} b_j$ with $\alpha_j, \overline{\alpha_j} \in \mathbb{F}$ for all $1 \leq i, j \leq r$. Note that the s_{ij}, s_{0i} and s_{i0} can be defined by the same expression for every mainline quotient of class $c \geq r$.

Proof:

This follows directly from the structure description in Theorem 1.4. Let $A \cong T \ltimes \text{Ann}_*(A)$ describe the infinite path in \mathcal{T} . Then A has a presentation of the form

$$\begin{aligned} \langle t, b_1, \dots, b_r \mid & b_i b_j - s_{ij}, \text{ for } 1 \leq i, j \leq r \\ & t b_i - s_{0i}, \text{ for } 1 \leq i \leq r, \\ & b_i t - s_{i0}, \text{ for } 1 \leq i \leq r \rangle \end{aligned}$$

where the s_{ij} are of the given form. The generator t is a generator for the infinite subalgebra $T \cong \mathbb{F}_o[[t]]$ and b_1, \dots, b_r correspond to a basis for $\text{Ann}_*(A)$. This directly implies that for $c \geq r$ the class c mainline algebra of \mathcal{T} can be described by a presentation of the given form. The generators for the mainline algebra correspond to the generators of A under the natural projection to A/A^{c+1} . \square

Lemma 4.9

Let \mathcal{T} be a maximal coclass tree in $\mathcal{G}_{\mathbb{F}}(r)$ and let $A_1 \rightarrow A_2 \rightarrow \dots$ be the unique infinite path starting at the root of \mathcal{T} . Let $B \in \mathcal{B}_k$ for some $k \in \mathbb{N}$ such that $c = \text{cl}(A_k) \geq r + 1$. Furthermore, let $d = \text{cl}(B) - c$. Then B has a presentation of the following form:

$$\begin{aligned} \langle t, b_1, \dots, b_r \mid & t^{c+d+1}, \\ & b_i b_j - s_{ij} - t_{ij}, \text{ for } 1 \leq i, j \leq r, \\ & t b_i - s_{0i} - t_{0i}, \text{ for } 1 \leq i \leq r, \\ & b_i t - s_{i0} - t_{i0}, \text{ for } 1 \leq i \leq r \rangle \end{aligned}$$

where t_{ij} lies in the subspace $\langle t^{c+1}, \dots, t^{c+d} \rangle$ for all possible i, j and the s_{ij} are the same expressions as in the presentation of the mainline algebra A_k .

Proof:

By Lemma 4.8 A_k has a presentation

$$\begin{aligned} \langle t, b_1, \dots, b_r \mid & t^{c+1}, \\ & b_i b_j - s_{ij}, \text{ for } 1 \leq i, j \leq r, \\ & t b_i - s_{0i}, \text{ for } 1 \leq i \leq r, \\ & b_i t - s_{i0}, \text{ for } 1 \leq i \leq r \rangle. \end{aligned}$$

Now consider an extension of A_k by a 1-dimensional trivial A_k -module $I = \langle x \rangle$. If we require the extension to be of the same coclass, it needs to be of class $c + 1$. Hence every immediate descendant of A_k in $\mathcal{G}_{\mathbb{F}}(r)$ has a presentation of the form

$$\begin{aligned} \langle t, b_1, \dots, b_r, c \mid & t^{c+1} - \beta_{00}x, \\ & b_i b_j - s_{ij} - \beta_{ij}x, \text{ for } 1 \leq i, j \leq r, \\ & t b_i - s_{0i} - \beta_{0i}x, \text{ for } 1 \leq i \leq r, \\ & b_i t - s_{i0} - \beta_{i0}x, \text{ for } 1 \leq i \leq r \rangle, \end{aligned}$$

where $\beta_{00} \in \mathbb{F}^*$ and all other $b_{ij} \in \mathbb{F}$. By rescaling we arrive at a presentation of the form

$$\begin{aligned} \langle t, b_1, \dots, b_r \mid & t^{c+2}, \\ & b_i b_j - s_{ij} - t_{ij}, \text{ for } 1 \leq i, j \leq r, \\ & t b_i - s_{0i} - t_{0i}, \text{ for } 1 \leq i \leq r, \\ & b_i t - s_{i0} - t_{i0}, \text{ for } 1 \leq i \leq r \rangle. \end{aligned}$$

with all possible t_{ij} being in the subspace $\langle t^{c+1} \rangle$ and the s_{ij} being the same expressions as in the presentation for the mainline algebra A_k . Iterating this yields the desired result. \square

Definition 4.10

Let B be as in Lemma 4.9. Then we call a presentation constructed as in Lemma 4.9 a **t -presentation**. We call such a t -presented B **t -split** if it is a split extension of the subalgebra generated by t and some ideal of B .

Remark 4.11

If we construct t -presentations for all algebras in \mathcal{T} that have the class $r+1$ mainline algebra as quotient, then such an algebra is t -split if and only if it is mainline.

In the next lemma we construct a special generating set for a t -presented algebra B . An immediate consequence of this is a sum decomposition of B into the subalgebra generated by t and the right annihilator of a certain t -power. This enables us to prove a depth bound for almost all branches of the maximal coclass tree \mathcal{T} .

Lemma 4.12

Let B be as in Lemma 4.9 and let its t -presentation have the form

$$\begin{aligned} \langle t, b_1, \dots, b_r \mid & t^{c+d+1}, \\ & b_i b_j - s_{ij} - t_{ij}^{(r+1)}, \text{ for } 1 \leq i, j \leq r, \\ & t b_i - s_{0i} - t_{0i}^{(r+1)}, \text{ for } 1 \leq i \leq r, \\ & b_i t - s_{i0} - t_{i0}^{(r+1)}, \text{ for } 1 \leq i \leq r \rangle \end{aligned}$$

Let \bar{s}_{0i} for $1 \leq i \leq r$ be defined as follows. If $s_{0i} = \sum_{j=i+1}^r \alpha_j b_j$, then let $\bar{s}_{0i} = \sum_{j=i+1}^r \alpha_j \bar{b}_j$.

Then B has a generating set $t, \bar{b}_1, \dots, \bar{b}_r$ such that $t \bar{b}_i - \bar{s}_{0i} = 0$ holds for all $1 \leq i \leq r$.

Proof:

We construct the new generating set from the original one. To facilitate this, we describe how to replace each generator b_i with a new generator \bar{b}_i . We need to work from back to front, i.e. we first replace b_r by a new generator \bar{b}_r , then replace b_{r-1} by \bar{b}_{r-1} and continue until we have replaced all generators b_i by the corresponding new generators \bar{b}_i .

Again, let T be the subalgebra of B generated by t and let σ_ℓ be the left shift

$$\sigma_\ell : T^2 \rightarrow T, \quad \sum_{j=2}^{c+d} \alpha_j t^j \mapsto \sum_{j=2}^{c+d} \alpha_j t^{j-1}$$

where $\alpha_j \in \mathbb{F}$.

Now suppose that we have replaced the generators b_{i+1}, \dots, b_r by the corresponding new generators $\bar{b}_{i+1}, \dots, \bar{b}_r$. To replace b_i we do the following.

- Define the new generator as $\bar{b}_i = b_i + \sigma_\ell(t_{0i}^{(i+1)})$.
- Adjust the relators accordingly, i.e. replace every occurrence of b_i in a relator by \bar{b}_i and replace the corresponding $t_{ij}^{(i+1)}$ by $t_{ij}^{(i)} = t_{ij}^{(i+1)} - \sigma_\ell(t_{0i}^{(i+1)})$.

After replacing all the original generators b_i by the corresponding \bar{b}_i we have constructed a presentation on a new generating set $t, \bar{b}_1, \dots, \bar{b}_r$ that has the desired property. \square

Example 4.13

We illustrate the construction of this special generating set in the following example, which is an \mathbb{F}_2 -algebra of coclass 2 and dimension 8.

- The algebra is given by the following t -presentation.

$$\langle t, a, b \mid t^7, a^2, ab, ba, b^2, at, bt, ta - b - t^6, tb - t^6 \rangle.$$

- Replace the generator b by $\bar{b} = b + \sigma_\ell(t^6) = b + t^5$ and get

$$\langle t, a, \bar{b} \mid t^7, a^2, a\bar{b}, \bar{b}a, \bar{b}^2, at, \bar{b}t - t^6, ta - \bar{b} - t^5 - t^6, t\bar{b} \rangle.$$

- Replace the generator a by $\bar{a} = a + \sigma_\ell(t^5 + t^6) = a + t^4 + t^5$ and get

$$\langle t, \bar{a}, \bar{b} \mid t^7, \bar{a}^2, \bar{a}\bar{b}, \bar{b}\bar{a}, \bar{b}^2, \bar{a}t - t^5 - t^6, \bar{b}t - t^6, t\bar{a} - \bar{b}, t\bar{b} \rangle.$$

The new generating set t, \bar{a}, \bar{b} has the desired property.

Corollary 4.14

Let B be as in Lemma 4.9. Then for some $u \leq r$ we have that $B = T + \text{Ann}_R(\{t^u\})$, where T is the subalgebra of B generated by t .

Proof:

This is a direct consequence of Lemma 4.12, as by construction the generators $\bar{b}_1, \dots, \bar{b}_r$ are contained in $\text{Ann}_R(\{t^u\})$ for some $u \leq r$. \square

Remark 4.15

If we take a closer look at the construction of the new generating set $t, \bar{b}_1, \dots, \bar{b}_r$ in Lemma 4.12 we notice the correspondence between the relations $tb_i - s_{0i} = 0$ in the t -presentation of the corresponding mainline algebra A_k and the relations $t\bar{b}_i - \bar{s}_{0i} = 0$. More explicitly, if the t -presentation for the corresponding mainline algebra contains the relation

$$tb_i - \sum_{j=i+1}^r \alpha_j b_j = 0$$

then the newly constructed generating set satisfies

$$t\bar{b}_i - \sum_{j=i+1}^r \alpha_j \bar{b}_j = 0.$$

Hence we can also read off the minimal u such that $B = T + \text{Ann}_R(\{t^u\})$ from the t -presentation of the corresponding mainline quotient. This in turn allows us to read off the minimal u from the presentation of the pro-nilpotent algebra describing the mainline of the coclass tree.

Theorem 4.16

Let \mathcal{T} be a maximal coclass tree in $\mathcal{G}_{\mathbb{F}}(r)$. Let $B \in \mathcal{B}_i$ with $\text{cl}(B) > 2r$ and let $u \leq r$ be minimal with $B = T + \text{Ann}_R(\{t^u\})$. Then $B/B^{\text{cl}(B)-u+1}$ is a mainline algebra of \mathcal{T} .

Proof:

First note that since $\text{cl}(B) > 2r$ and $u \leq r$, we have $B^{\text{cl}(B)-u+1} = T^{\text{cl}(B)-u+1}$. Also because $\text{cl}(B) > 2r$ the quotient $B/B^{\text{cl}(B)-u+1}$ also has coclass r . The right annihilator $\text{Ann}_R(\{t^u\})$ is obviously a right ideal, but it also is a left ideal as we show now. Let $b \in B$ and let $z \in \text{Ann}_R(\{t^u\})$. Write $b = x + y$ with $x \in T$ and $y \in \text{Ann}_R(\{t^u\})$. Then $bz = xz + yz$. Trivially $yz \in \text{Ann}_R(\{t^u\})$. Now x commutes with t^u because $x \in T$. Therefore $t^u xz = xt^u z = 0$ and $xz \in \text{Ann}_R(\{t^u\})$ follows. In conclusion $bz \in \text{Ann}_R(\{t^u\})$ and $\text{Ann}_R(\{t^u\})$ is also a left ideal, hence an ideal in B . Observe that the only t -powers contained in $\text{Ann}_R(\{t^u\})$ are $t^{\text{cl}(B)-u+1}, \dots, t^{\text{cl}(B)}$ and thus $T^{\text{cl}(B)-u+1}$ is an ideal contained in T and $\text{Ann}_R(\{t^u\})$. Furthermore, we have

$$B/B^{\text{cl}(B)-u+1} = (T + \text{Ann}_R(\{t^u\}))/T^{\text{cl}(B)-u+1} \cong T/T^{\text{cl}(B)-u+1} \ltimes \text{Ann}_R(\{t^u\})/T^{\text{cl}(B)-u+1}.$$

If the t -presentation of B is given by

$$\begin{aligned} \langle t, b_1, \dots, b_r \mid & t^{\text{cl}(B)+1}, \\ & b_i b_j - s_{ij} - t_{ij}, \text{ for } 1 \leq i, j \leq r \text{ and } s_{ij} \in \langle b_1, \dots, b_r \rangle, \\ & tb_i - s_{0i} - t_{0i}, \text{ for } 1 \leq i \leq r, \\ & b_i t - s_{i0} - t_{i0}, \text{ for } 1 \leq i \leq r \rangle \end{aligned}$$

then a presentation for $B/B^{\text{cl}(B)-u+1}$ is given by

$$\begin{aligned} \langle t, b_1, \dots, b_r \mid & t^{\text{cl}(B)-u+1}, \\ & b_i b_j - s_{ij} - t_{ij}, \text{ for } 1 \leq i, j \leq r \text{ and } s_{ij} \in \langle b_1, \dots, b_r \rangle, \\ & tb_i - s_{0i} - t_{0i}, \text{ for } 1 \leq i \leq r, \\ & b_i t - s_{i0} - t_{i0}, \text{ for } 1 \leq i \leq r \rangle \end{aligned}$$

which is precisely the t -presentation for the ancestor of B of distance u . By the above calculation it is t -split, hence a mainline algebra by Remark 4.11. \square

In conclusion, we can now deduce the Theorem 1.11 for maximal coclass trees in $\mathcal{G}_{\mathbb{F}}(r)$.

Theorem 1.11

Let \mathbb{F} be an arbitrary field and let r be a non-negative integer. Let \mathcal{T} be a maximal coclass tree in $\mathcal{G}_{\mathbb{F}}(r)$. Then \mathcal{T} has bounded depth.

Proof:

Let \mathcal{B}_i be a branch in \mathcal{T} . If $cl(A_i) \geq 2r$, then Theorem 4.16 implies that $dep(\mathcal{B}_i) \leq r$. The finitely many branches \mathcal{B}_i with $cl(A_i) < 2r$ have finite depth because \mathcal{T} is a maximal coclass tree, i.e. \mathcal{T} contains a unique infinite path starting at the root of \mathcal{T} . Therefore \mathcal{T} has bounded depth. \square

Remark 4.17

This is stronger than the corresponding result in the p -group case. For p -groups the maximal coclass trees in $\mathcal{G}(p, r)$ have bounded depth if and only if $p = 2$ or $(p, r) = (3, 1)$ (see e.g. Leedham-Green & McKay [LGM02]). The simplest case with unbounded depth, i.e. $(p, r) = (5, 1)$, is investigated in detail in a paper by Dietrich [Die10]. In this sense the graphs for nilpotent associative \mathbb{F} -algebras behave similar to the ones for 2-groups.

Remark 4.18

From the proof of theorem 1.11 one can see that \mathcal{T} virtually has depth less or equal than r . Note that in all of our computational data, the bound holds for all branches, i.e. $dep(\mathcal{B}_i) \leq r$ for all $i \in \mathbb{N}$. Thus conjecturally the stronger statement $dep(\mathcal{T}) \leq r$ holds.

Remark 4.19

In many cases the depth bound obtained from Theorem 4.16 is even smaller than r . Furthermore, the same argument works using left annihilators instead of right annihilators. The depth bounds obtained from Theorem 4.16 and its analogue for left annihilators can differ. If one is interested in the best bound that can be obtained in this way, one should consider both types of annihilators.

Example 4.20

To illustrate Remark 4.19 we consider the following pro-nilpotent \mathbb{F}_2 -algebra describing an infinite path of a maximal coclass tree \mathcal{T} in $\mathcal{G}_{\mathbb{F}_2}(r)$.

$$\begin{aligned} \langle t, b_1, \dots, b_r \mid & b_i b_j, & \text{for } 1 \leq i, j \leq r \\ & t b_i, & \text{for } 1 \leq i \leq r \\ & b_i t - b_{i+1}, & \text{for } 1 \leq i \leq r-1 \\ & b_r t \rangle \end{aligned}$$

The depth bound we obtain from right annihilators is $dep(\mathcal{B}) \leq 1$ for almost all branches \mathcal{B} of \mathcal{T} , whereas the depth bound we obtain from left annihilators is $dep(\mathcal{B}) \leq r$ for almost all branches \mathcal{B} of \mathcal{T} . This shows that in some cases the bound can be improved significantly by considering both types of annihilators.

4.3 General periodicity conjecture

As already mentioned in the introduction we conjecture a certain form of periodicity for the maximal coclass trees in $\mathcal{G}_{\mathbb{F}}(r)$. We recall the definition of virtual periodicity, which precisely describes the type of periodicity we conjecture.

Definition 1.12

Let \mathbb{F} be an arbitrary field and let r be a non-negative integer. Let \mathcal{T} be a maximal coclass tree in $\mathcal{G}_{\mathbb{F}}(r)$ with unique infinite path $A_1 \rightarrow A_2 \rightarrow \dots$ starting at the root of \mathcal{T} . Then \mathcal{T} is called **virtually periodic** with **period** d and **periodic root** A_ℓ if the descendant trees $\mathcal{T}(A_i)$ and $\mathcal{T}(A_{i+d})$ are isomorphic as directed graphs for each $i \geq \ell$.

Based on computational evidence we stated in [EM16] the following general conjecture for finite fields.

Conjecture 1.13

Let \mathbb{F} be an arbitrary finite field and let r be a non-negative integer and let \mathcal{T} be maximal coclass tree in $\mathcal{G}_{\mathbb{F}}(r)$. Then \mathcal{T} is virtually periodic.

Remark 4.21

Our experimental evidence suggests that if \mathcal{T} is a maximal coclass tree in $\mathcal{G}_{\mathbb{F}}(r)$ and \mathbb{F} is a field of size $q = p^n$, then it has a period of the form $p^s(q-1)$ with $s \leq r$.

This is similar to a periodicity result for the coclass trees associated with finite 2-groups (see e.g. Eick & Leedham-Green [ELG08]). The proof of the result in [ELG08] also implies that the 2-groups of a fixed coclass r can be described by finitely many parametrized presentations. Again in [EM16] we conjectured a similar result for nilpotent associative \mathbb{F} -algebras over finite fields.

Conjecture 1.14

Let \mathbb{F} be an arbitrary finite field and let r be a non-negative integer and let \mathcal{T} be maximal coclass tree in $\mathcal{G}_{\mathbb{F}}(r)$. Then the algebras in \mathcal{T} can be described by finitely many parametrized presentations.

These conjectures obviously hold for $r = 0$. In the following we prove Conjectures 1.13 and 1.14 for $r = 1$. In the case of $r = 2$ we give a very detailed conjecture.

4.4 Periodicity for coclass 1

In order to prove the conjectures mentioned above for coclass 1, we introduce a machinery that facilitates the computation of immediate descendants. We use concepts developed by Eick [Eic08], which we recall here. Let \mathbb{F} be a finite field and let A be a nilpotent associative \mathbb{F} -algebra. Let $n = \dim(A)$, $c = \text{cl}(A)$ and let $d = \dim(A/A^2)$ be the minimal generator number of A . We denote by F the free non-unital \mathbb{F} -algebra on d generators. Then there is an ideal R , such that $A \cong F/R$.

Definition 4.22

- The **covering algebra** A^* of A is defined as $A^* = F/(FR \cup RF)$, where $(FR \cup RF)$ is the ideal generated by $FR \cup RF$ in F .
- The **multiplicator** of A is defined as $M(A) = R/(FR \cup RF)$.
- The **nucleus** $N(A)$ of A is defined as $(A^*)^{c+1} \leq M(A)$.

- A proper subspace $U < M(A)$ is called **allowable** if it supplements the nucleus, i.e. $U + N(A) = M(A)$.

In [Eic08] it is shown that $M(A)$ is a finite-dimensional \mathbb{F} -vector space. Furthermore, it is shown that the natural homomorphism $\epsilon : \text{Aut}(A^*) \rightarrow \text{Aut}(A)$ is surjective and that its kernel consists of those automorphisms that act trivially on the multiplier $M(A)$. Hence we obtain an action of $\text{Aut}(A)$ on $M(A)$ and hence on the allowable subspaces of $M(A)$ via ϵ . We can deduce the following theorem from [Eic08].

Theorem 4.23

Let \mathfrak{L} denote a set of orbit representatives of the action of $\text{Aut}(A)$ on the allowable subspaces of $M(A)$. Then

$$\{A^*/U \mid U \in \mathfrak{L}\}$$

is a complete and irredundant set of isomorphism type representatives of immediate descendants of A .

Proof:

By [Eic08, Theorem 7] every immediate descendant of A is a quotient of A^* by some allowable subspace and, conversely, every quotient of the covering algebra A^* by an allowable subspace yields an immediate descendant of A . Now [Eic08, Theorem 10 (a)] states that two quotients of A^* by allowable subspaces yield isomorphic immediate descendants if and only if the allowable subspaces are in the same orbit under the action of $\text{Aut}(A)$. \square

Theorem 4.23 has the following immediate corollary, which allows to decide capability by calculating the nucleus.

Corollary 4.24

A finite-dimensional nilpotent associative \mathbb{F} -algebra is capable if and only if the nucleus $N(A)$ is non-trivial.

Proof:

This is a direct consequence of Theorem 4.23, because there are allowable subspaces if and only if $\dim(N(A)) > 0$. \square

We now turn to the proof of Conjectures 1.13 and 1.14 for coclass 1. For $r = 1$ we can deduce from Corollary 3.38 that the coclass graph $\mathcal{G}_{\mathbb{F}}(1)$ consists of a single maximal coclass tree $\mathcal{T}(\mathcal{Z}_2)$. Let $\mathcal{Z}_2 = A_1 \rightarrow A_2 \rightarrow \dots$ be the infinite path. Then the mainline algebra A_i has class i and can be presented as

$$A_i \cong \langle t, a \mid t^{i+1}, a^2, ta, at \rangle.$$

By Theorem 4.16 the branches \mathcal{B}_i for $i \geq 2$ have depth at most 1. We proceed by explicitly doing the descendant calculation described in Algorithm 5.1 for the mainline algebras A_i with $i \geq 2$.

Lemma 4.25

For $i \geq 2$ the covering algebra A_i^* of the mainline algebra A_i is given by

$$A_i^* \cong \langle t, a \mid t^{i+2}, ta^2, t^2a, tat, a^2t, at^2, a^3, ata \rangle.$$

The multiplier $M(A_i)$ is 4-dimensional and has a basis given by $\{t^{i+1}, a^2, ta, at\}$. The nucleus $N(A_i)$ is 1-dimensional and has a basis given by $\{t^{i+1}\}$.

Proof:

This follows directly from the definitions of the covering algebra, multiplier and nucleus. \square

By the previous Lemma 4.25 we see that for all $i \geq 2$ the multiplier $M(A_i)$ is 4-dimensional and the nucleus is 1-dimensional. Hence in all cases the number of immediate descendants is equal to the number of orbits of $W_i = \{\alpha \in \text{Aut}(A_i^*) \mid \alpha(M(A_i)) = M(A_i)\}$ on the set of supplements to the 1-dimensional subspaces $N(A_i)$ in $M(A_i)$. In the next lemma we describe the action of W_i on $M(A_i)$ and obtain that the actions of W_i and $W_{i+|\mathbb{F}|-1}$ on the respective multipliers coincide.

Lemma 4.26

The action of W_i on $M(A_i)$ can be described as follows.

$$\begin{aligned} t^{i+1} &\mapsto \alpha_1^{i+1} t^{i+1} \\ a^2 &\mapsto \bar{\beta} a^2 \\ ta &\mapsto \alpha_1 \bar{\alpha}_i t^{i+1} + \beta \bar{\beta} a^2 + \alpha_1 \bar{\beta} ta \\ at &\mapsto \alpha_1 \bar{\alpha}_i t^{i+1} + \beta \bar{\beta} a^2 + \alpha_1 \bar{\beta} at \end{aligned}$$

where $\alpha_1, \bar{\beta} \in \mathbb{F}^*$ and $\bar{\alpha}_i, \beta \in \mathbb{F}$.

Proof:

We explicitly determine the groups W_i . Consider a general map $\varphi : A_i^* \rightarrow A_i^*$ given on the generators t and a by

$$\begin{aligned} t &\mapsto \beta a + \sum_{j=1}^{i+1} \alpha_j t^j + \gamma a^2 + \delta ta + \epsilon at \\ a &\mapsto \bar{\beta} a + \sum_{j=1}^{i+1} \bar{\alpha}_j t^j + \bar{\gamma} a^2 + \bar{\delta} ta + \bar{\epsilon} at \end{aligned}$$

We want to filter out those maps that are automorphisms fixing the multiplier setwise. First note that $\bar{\alpha}_1 = 0$, because otherwise $a^2 \in M(A_i)$ would not get mapped into $M(A_i)$. As φ needs to be surjective, we have $\alpha_1 \neq 0$ and $\bar{\beta} \neq 0$. Next we have to ensure that the relators of the covering algebra get mapped into the relators of the covering algebra. Consider the relator t^2a .

$$t^2a \mapsto \sum_{n=4}^{i+1} \sum_{j+k+\ell=n} \alpha_j \alpha_k \bar{\alpha}_\ell t^n, \text{ where } j, k \in \{1, \dots, i+1\}, \ell \in \{2, \dots, i+1\}.$$

The coefficient of t^4 is $\alpha_1^2 \overline{\alpha_2}$ and as $\alpha_1 \neq 0$ it follows that $\overline{\alpha_2} = 0$. Now considering the coefficients for t^5, \dots yields $\overline{\alpha_2} = \dots = \overline{\alpha_{i-1}} = 0$. One checks that mapping the other relators does not give additional restrictions on φ and that indeed all φ defined in this way are automorphisms of A_i^* mapping $M(A_i)$ into itself. Explicitly writing down the action of φ on the multiplier yields

$$\begin{aligned} t^{i+1} &\mapsto \varphi(t^{i+1}) = \varphi(t)^{i+1} = \alpha_1^{i+1} t^{i+1} \\ a^2 &\mapsto \varphi(a^2) = \varphi(a)^2 = \overline{\beta} a^2 \\ ta &\mapsto \varphi(ta) = \varphi(t)\varphi(a) = \alpha_1 \overline{\alpha_i} t^{i+1} + \beta \overline{\beta} a^2 + \alpha_1 \overline{\beta} ta \\ at &\mapsto \varphi(at) = \varphi(a)\varphi(t) = \alpha_1 \overline{\alpha_i} t^{i+1} + \beta \overline{\beta} a^2 + \alpha_1 \overline{\beta} at \end{aligned}$$

where $\alpha_1, \overline{\beta} \in \mathbb{F}^*$ and $\overline{\alpha_i}, \beta \in \mathbb{F}$ as desired. \square

Corollary 4.27

For $i \geq 2$ the action of W_i on $M(A_i)$ coincides with the action of $W_{i+|\mathbb{F}|-1}$ on $M(A_{i+|\mathbb{F}|-1})$.

Proof:

Note that the coefficient α_1^{i+1} is the only one really depending on i . Furthermore, an element $\gamma \in \mathbb{F}$ is an $(i+1)$ -th power if and only if it is an $(i+|\mathbb{F}|)$ -th power, as $\gamma^{|\mathbb{F}|-1} = 1$ holds for all elements in a finite field. This proves the claim. \square

Corollary 4.28

The single maximal coclass tree $\mathcal{T}(\mathcal{Z}_2)$ in $\mathcal{G}_{\mathbb{F}}(1)$ has depth 1 and is virtually periodic with period $|\mathbb{F}| - 1$ and periodic root A_2 .

Proof:

Recall that the proof of Theorem 1.11 implies that the branches \mathcal{B}_i for $i \geq 2$ have depth at most 1. From Corollary 4.27 we now deduce that there is an isomorphism of directed graphs $\mathcal{B}_i \cong \mathcal{B}_{i+|\mathbb{F}|-1}$ for all $i \geq 2$. It remains to show that the branch \mathcal{B}_1 has depth 1. This can be done by a direct calculation. \square

The above Corollary 4.28 is the desired positive answer for Conjectures 1.13 and 1.14 in the case of coclass 1. Unfortunately, it seems very hard to translate this approach of explicit calculations to the general case. However, it seems plausible to extend this method to the case of a pruned coclass tree $\mathcal{T}^{(1)}$, where one takes a coclass tree \mathcal{T} and cuts off all vertices of distance larger than 1 from the mainline. This would then prove the virtual periodicity for all coclass trees of depth 1, which e.g. for coclass 2 turn out to be all trees except one. As an example we give a classification of all algebras in $\mathcal{G}_{\mathbb{F}_2}(1)$, before giving a detailed conjecture for the shape of $\mathcal{G}_{\mathbb{F}}(2)$.

Example 4.29

The d -dimensional nilpotent associative \mathbb{F}_2 -algebras of coclass 1 can be fully described as follows.

- $d = 2$ and $d = 3$ are known by the general classification mentioned in the introduction.

- $d \geq 4$:

These algebras are the immediate descendants of the mainline algebra $A_{d-1,1} = A_{d-2}$. We explicitly do the orbit calculation suggested by Lemma 4.25 and Lemma 4.26. By choosing orbit representatives we get the following d -dimensional nilpotent associative \mathbb{F}_2 -algebras of coclass 1 for $d \geq 4$ and hence obtain a full classification of the algebras in $\mathcal{G}_{\mathbb{F}_2}(1)$.

$$A_{d,1} = A_{d-1} = \langle t, a \mid t^d, a^2, ta, at \rangle \text{ (mainline algebra)}$$

$$A_{d,2} = \langle t, a \mid t^d, a^2 - t^{d-1}, ta, at \rangle$$

$$A_{d,3} = \langle t, a \mid t^d, a^2, ta - t^{d-1}, at \rangle$$

$$A_{d,4} = \langle t, a \mid t^d, a^2 - t^{d-1}, ta - t^{d-1}, at \rangle$$

Remark 4.30

The reasoning in the case of coclass 1 can also be used to obtain results for certain infinite fields, e.g. $\mathcal{G}_{\mathbb{R}}(1)$ will be virtually periodic with period at most 2 and $\mathcal{G}_{\mathbb{C}}(1)$ will be virtually periodic with period 1.

4.5 Periodicity conjecture for coclass 2

The following conjecture (see our paper [EM16, Conjecture 5]) describes the shape of $\mathcal{G}_{\mathbb{F}}(2)$ for finite fields in detail. It is based on our experimental data.

Conjecture 4.31

Let \mathbb{F} be a finite field of characteristic p with q elements.

- (1) Let $p = 2$. Then $\mathcal{G}_{\mathbb{F}}(2)$ consists of $3q + 5$ maximal descendant trees. Of these trees $2q + 1$ are finite and $q + 4$ are maximal coclass trees. The maximal coclass trees are all virtually periodic and have finite depth. More precisely:
 - (a) There are $q + 3$ maximal coclass trees of depth 1 and period $q - 1$.
 - (b) There is 1 maximal coclass tree of depth 2 and period $p(q - 1)$.
- (2) Let $p > 2$. Then $\mathcal{G}_{\mathbb{F}}(2)$ consists of $3q + 6$ maximal descendant trees. Of these trees $2q + 2$ are finite and $q + 4$ are maximal coclass trees. The maximal coclass trees are all virtually periodic and have finite depth. More precisely:
 - (a) There are $q + 1$ maximal coclass trees of depth 1 and period $q - 1$.
 - (b) There is 1 maximal coclass tree of depth 1 and period 1.
 - (c) There are 2 maximal coclass trees of depth 2 and period $p(q - 1)$.

Remark 4.32

In Conjecture 4.31 the number of maximal coclass trees is indeed a fact proven by Theorem 3.44. Theorem 4.16 yields the conjectured depth bound for almost all branches.

Chapter 5

Algorithms

5.1 Algorithm to compute immediate descendants

The aim of this section is to explicitly write down an algorithm for computing immediate descendants of a nilpotent associative \mathbb{F} -algebra A for finite fields \mathbb{F} . More precisely we determine a complete and irredundant list of isomorphism type representatives of immediate descendants of A . A preliminary version of this algorithm has been used by Bertram [Ber11]. It is based on Theorem 4.23, which translates naturally to the following algorithm for determining immediate descendants of a nilpotent associative \mathbb{F} -algebra up to isomorphism. This algorithm is also contained in our paper [EM16].

Algorithm 5.1 (Immediate descendants)

Input : *A finite-dimensional nilpotent associative \mathbb{F} -algebra over a finite field \mathbb{F} described by structure constants*

Output: *A set of isomorphism type representatives of immediate descendants of A described by structure constants*

Determine structure constants for A^ ;*
Compute $\text{Aut}(A)$ and its action on $M(A)$;
Build a list \mathfrak{S} of allowable subspaces $U < M(A)$;
Determine the orbits of $\text{Aut}(A)$ on \mathfrak{S} ;
foreach orbit do
 Choose a representative U ;
 Determine structure constants for $Q = A^/U$;*
 Store Q in a list X ;
end
Return the resulting list X ;

Remark 5.2

Methods for determining a structure constants table for A^ , computing the automorphism group of a nilpotent associative \mathbb{F} -algebra and its action of $M(A)$ are described by Eick [Eic08] and implemented in the GAP4 package modisom [Eic16].*

Remark 5.3

If one is only interested in the immediate descendants of a certain stepsize s , one only needs to calculate the allowable subspaces U with $\dim(U) = \dim(M(A)) - s$.

Remark 5.4

As described in [Eic08] the automorphism group of an immediate descendant Q can be determined from $\text{Stab}_{\text{Aut}(A)}(U)$. This proves to be extremely useful if the algorithm is applied iteratively, because the computation of the automorphism group can be skipped.

Remark 5.5

Starting with the d -generator algebras with trivial multiplication Z_d , the Algorithm 5.1 can be applied iteratively to determine the nilpotent associative \mathbb{F} -algebras of dimension n and rank d up to isomorphism. This was also used by Bertram in [Ber11] to calculate the nilpotent associative \mathbb{F}_2 -algebras of dimension 5.

Remark 5.6

If one is only interested in counting the number of nilpotent associative \mathbb{F} -algebras of dimension n , one can use the well-known Burnside/Cauchy-Frobenius lemma together with the methods developed by Eick and O'Brien in Section 4.2 of [EO99]. Note that one still has to calculate the algebras of dimension $n - 1$ in this case. Below is a table containing some data for small finite fields and small dimensions.

	\mathbb{F}_2	\mathbb{F}_3	\mathbb{F}_4	\mathbb{F}_5	\mathbb{F}_7	\mathbb{F}_8	\mathbb{F}_9
dim 4	27	35	37	45	55	57	65
dim 5	404	1758	6744	22117	144967		
dim 6	43049						

Number of isomorphism types of nilpotent associative \mathbb{F} -algebras

Remark 5.7

It is shown in Theorem 5.2.11 of [KP69] that the number of isomorphism types of nilpotent associative \mathbb{F}_q -algebras of dimension n is of the form $q^{\alpha n^3}$ where $\alpha = \frac{4}{27} + O(n^{-\frac{1}{3}})$.

5.2 Algorithm to calculate roots of maximal descendant trees

This section contains an algorithm to compute the roots of all maximal descendant trees in $\mathcal{G}_{\mathbb{F}}(r)$. As the case $r = 0$ is fully known, we only consider the case $r > 0$ here. Of course the main ingredients here are Algorithm 5.1 and Theorem 4.1. The latter theorem allows us to skip the calculation of immediate descendants of stepsize 1 when computing roots. The algorithm is also described in our paper [EM16].

Algorithm 5.8 (Roots of maximal descendants trees in $\mathcal{G}_{\mathbb{F}}(r)$ of rank d)

Input : A finite field \mathbb{F} and a rank $d \in \{2, \dots, r+1\}$

Output: The set of roots of maximal descendant trees in $\mathcal{G}_{\mathbb{F}}(r)$ that have rank d

Initialize lists X_d, \dots, X_{2r-2} and Y ;

Define A to be \mathcal{Z}_d , the d -dimensional algebra with trivial multiplication;

if $d = r + 1$ **then**

 | Add A to Y ;

else

 | Add A to X_d ;

end

for i **from** d **to** $2r - 2$ **do**

foreach $A \in X_i$ **do**

 | *Compute the immediate descendants of A of stepsize $s \in \{2, \dots, r - cc(A)\}$ and add these to X_{i+s} ;*

 | *Compute the immediate descendants of A of stepsize $r - cc(A) + 1$ and add these to Y ;*

end

end

Return Y ;

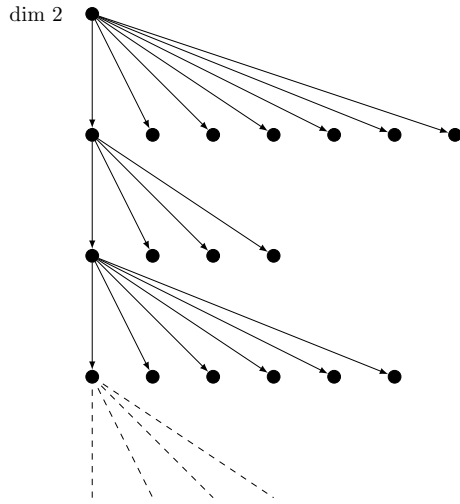
With Algorithm 5.1 and Algorithm 5.8 at hand, it is easy to calculate finite parts of a coclass graph $\mathcal{G}_{\mathbb{F}}(r)$ over a finite field. As a first step we calculate the roots of maximal descendant trees in $\mathcal{G}_{\mathbb{F}}(r)$ using the latter algorithm and then iteratively apply the first algorithm to calculate immediate descendants of stepsize 1, i.e. immediate descendants of the same coclass. In the following Chapter 6 we exhibit various examples of finite parts of coclass graphs for small finite fields and small coclasses. The implementation of the algorithms is in GAP [GAP15]. The first version of the corresponding package **ccalgs** is available at [EM15a].

Chapter 6

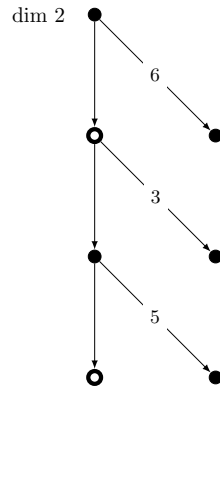
Applications of the algorithms

6.1 The coclass graph $\mathcal{G}_{\mathbb{F}}(1)$ with $|\mathbb{F}| < 20$

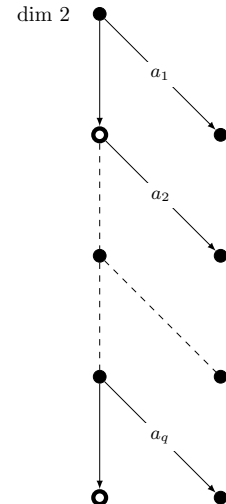
As a first application we describe the graphs $\mathcal{G}_{\mathbb{F}}(1)$ for all finite fields of order less than 20 in full detail. By Corollary 4.28 we know that these graphs consist of a single maximal coclass tree of depth 1 with infinite path $A_1 \rightarrow A_2 \rightarrow \dots$. It is virtually periodic with period $|\mathbb{F}| - 1$ and periodic root A_2 . Hence it suffices to use the algorithms for the calculation of the branches $\mathcal{B}_1, \dots, \mathcal{B}_{|\mathbb{F}|}$. We draw trees in a compacted form. Dots indicate vertices and the edges are numbered according to their multiplicity. The circles indicate the periodicity, meaning that the first circle is being mapped onto the second and this is repeated infinitely often. We illustrate this compacted form on the graph $\mathcal{G}_{\mathbb{F}_3}(1)$ that we have already presented in the introduction. In general, the graphs $\mathcal{G}_{\mathbb{F}_q}(1)$ are fully described by the values a_1, \dots, a_q . We give the calculated values for the finite fields of order less than 20 on the next page.



$\mathcal{G}_{\mathbb{F}_3}(1)$



$\mathcal{G}_{\mathbb{F}_3}(1)$ compacted



$\mathcal{G}_{\mathbb{F}}(1)$

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}	a_{19}
\mathbb{F}_2	4	3																	
\mathbb{F}_4	6	3	3	5															
\mathbb{F}_8	10	3	3	3	3	3	3	9											
\mathbb{F}_{16}	18	3	3	5	3	7	5	3	3	5	7	3	5	3	3	17			
\mathbb{F}_3	6	3	5																
\mathbb{F}_9	12	3	5	3	7	3	5	3	11										
\mathbb{F}_5	8	3	5	3	7														
\mathbb{F}_7	10	3	5	5	5	3	9												
\mathbb{F}_{11}	14	3	5	3	5	7	5	3	5	3	13								
\mathbb{F}_{13}	16	3	5	5	7	3	9	3	7	5	7	3	15						
\mathbb{F}_{17}	20	3	5	3	7	3	5	3	11	3	5	3	7	3	5	3	19		
\mathbb{F}_{19}	22	3	5	5	5	3	9	3	5	11	5	3	9	3	5	5	5	3	21

 Description of branches of $\mathcal{G}_{\mathbb{F}_q}(1)$

6.2 The coclass graph $\mathcal{G}_{\mathbb{F}}(2)$ with $|\mathbb{F}| \leq 5$

In this section we present the maximal coclass trees of the graphs $\mathcal{G}_{\mathbb{F}}(2)$ for all finite fields with $|\mathbb{F}| \leq 5$. In Section 3.3 it is shown that there are $|\mathbb{F}| + 4$ maximal coclass trees. The pro-nilpotent \mathbb{F} -algebras describing the infinite paths of these maximal coclass trees have presentations of the form

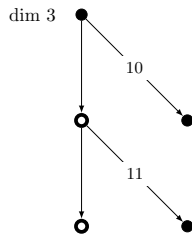
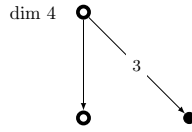
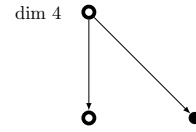
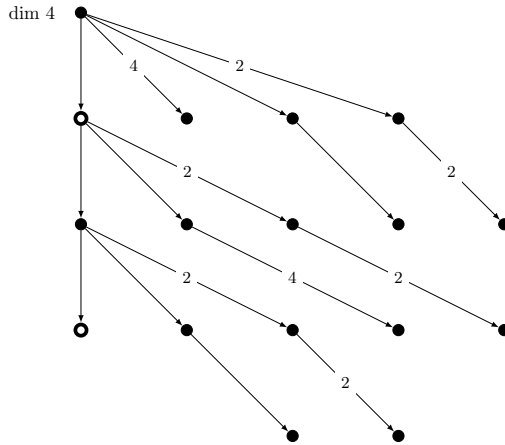
$$\langle t, a, b \mid at - \alpha_1 b, ta - \alpha_2 b, a^2 - \alpha_3 b, ab, ba, bt, tb, b^2 \rangle.$$

It was shown in Lemma 3.43 that a complete and irredundant set of isomorphism type representatives is given by the algebras described by the triples

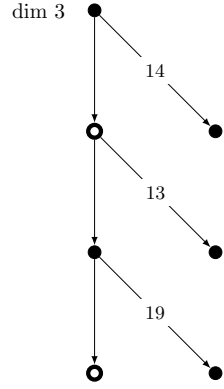
$$(\alpha_1, \alpha_2, \alpha_3) \in \{(0, 0, 0), (0, 0, 1), (0, 1, 1), (0, 1, 0), (1, x, 0) \mid x \in \mathbb{F}\}.$$

We denote the corresponding maximal coclass trees by $\mathcal{T}_{(\alpha_1, \alpha_2, \alpha_3)}$.

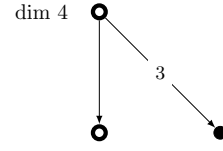
Maximal coclass trees in $\mathcal{G}_{\mathbb{F}_2}(2)$


 $\mathcal{T}_{(0,0,0)}$

 $\mathcal{T}_{(0,0,1)}, \mathcal{T}_{(0,1,0)}, \mathcal{T}_{(1,0,0)}$

 $\mathcal{T}_{(0,1,1)}$

 $\mathcal{T}_{(1,1,0)}$

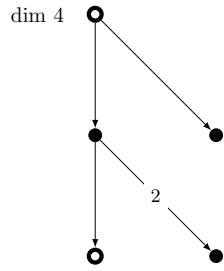
Maximal coclass trees in $\mathcal{G}_{\mathbb{F}_3}(2)$



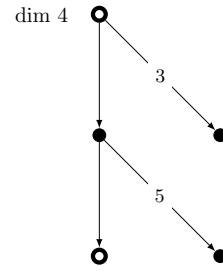
$\mathcal{T}_{(0,0,0)}$



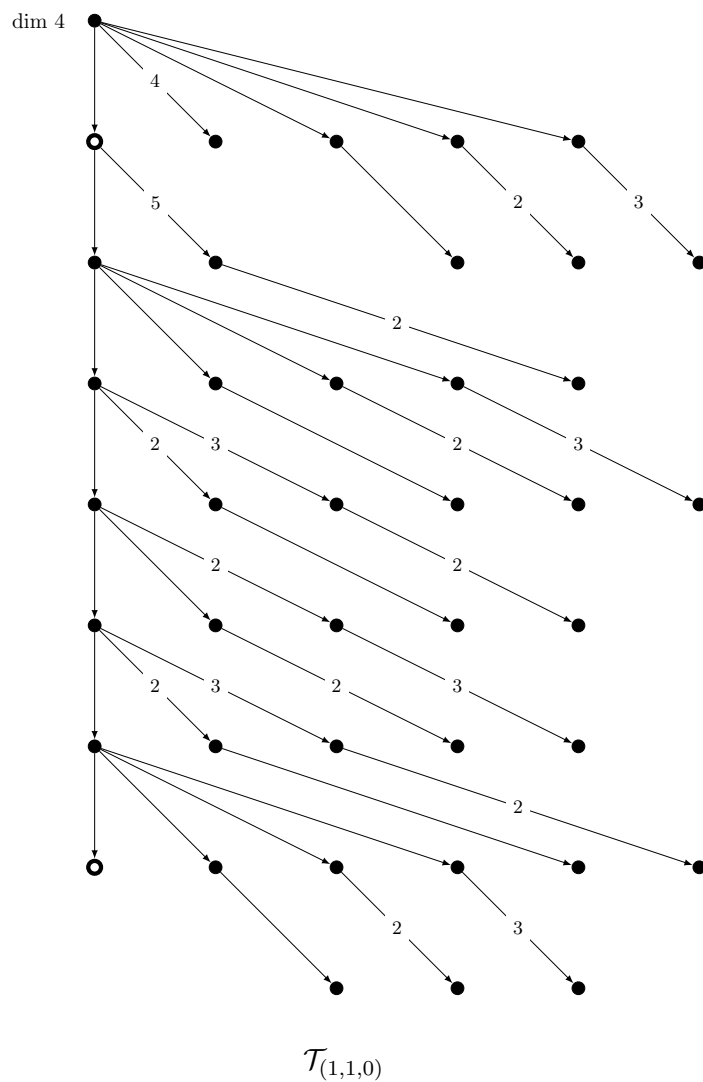
$\mathcal{T}_{(0,0,1)}$

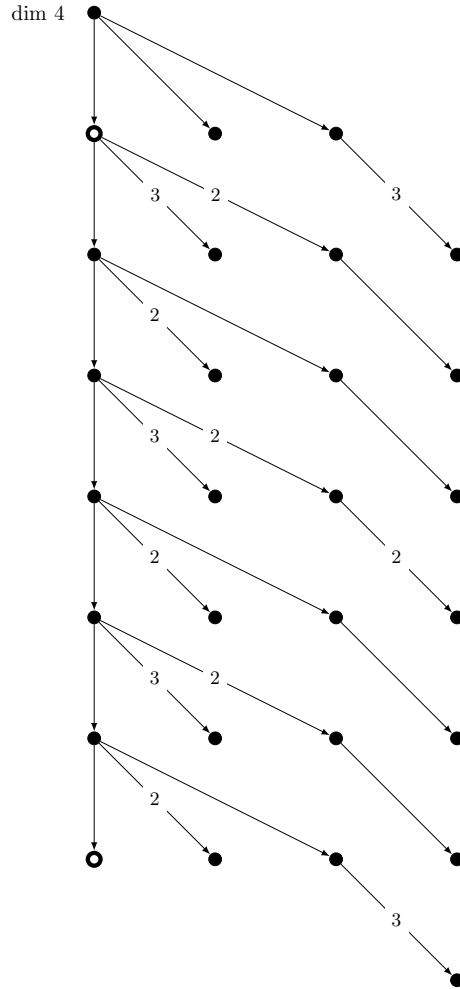


$\mathcal{T}_{(0,1,1)}$



$\mathcal{T}_{(0,1,0)}, \mathcal{T}_{(1,0,0)}$

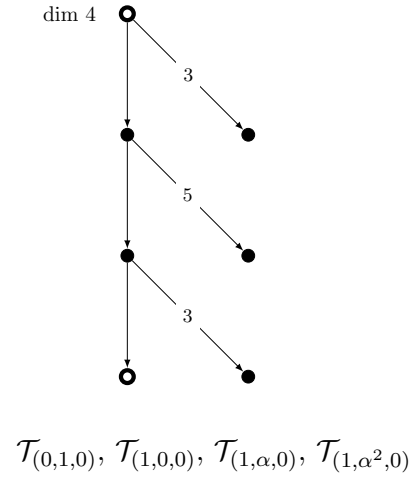
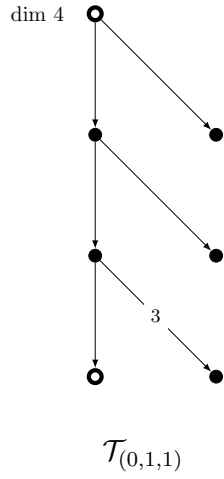
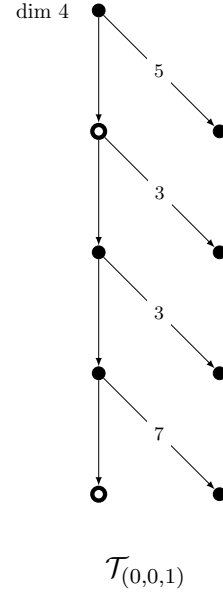
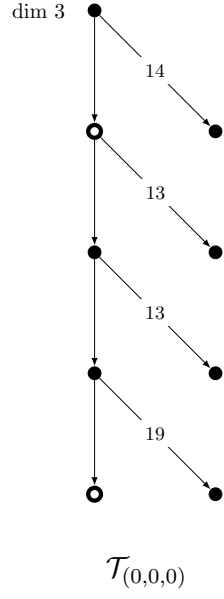


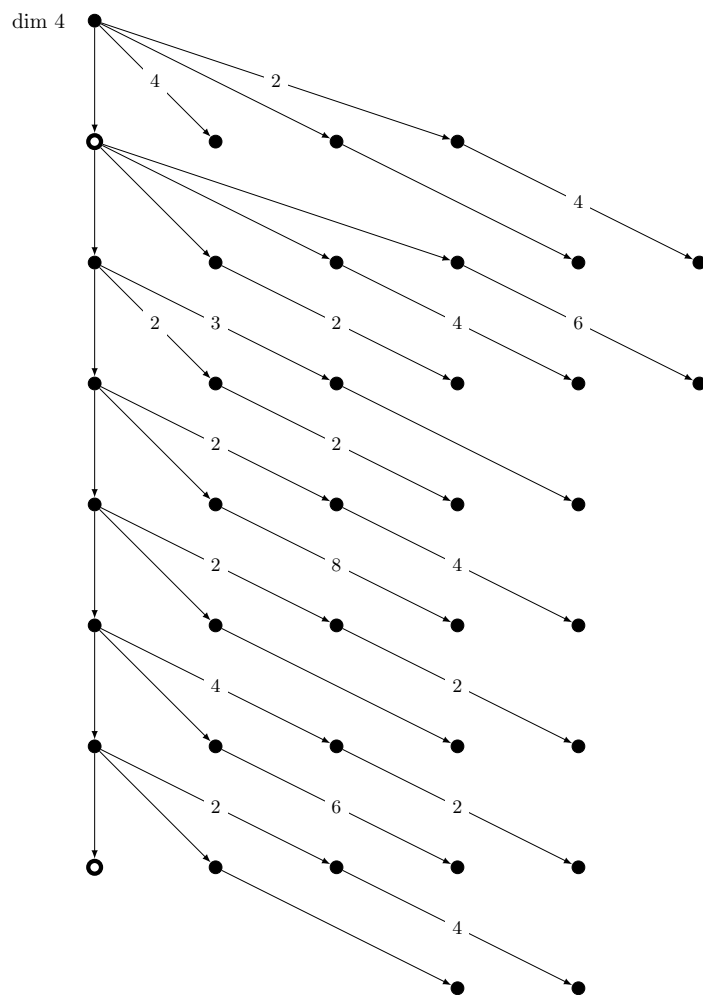


$\mathcal{T}_{(1,2,0)}$

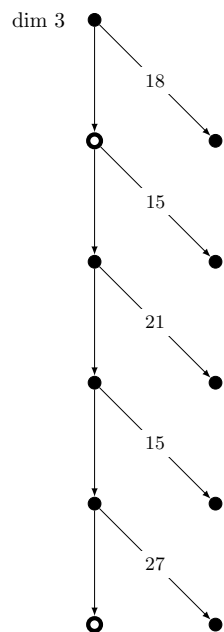
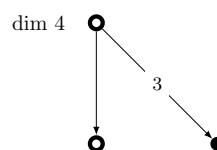
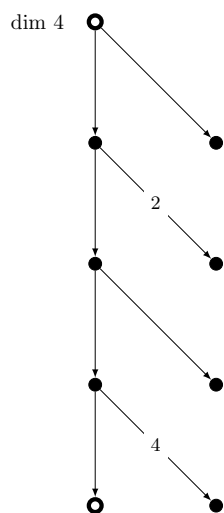
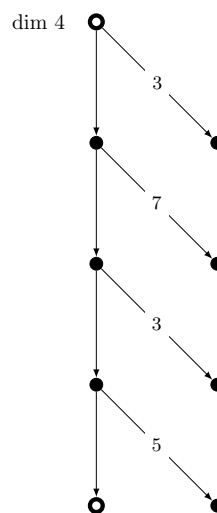
Maximal coclass trees in $\mathcal{G}_{\mathbb{F}_4}(2)$

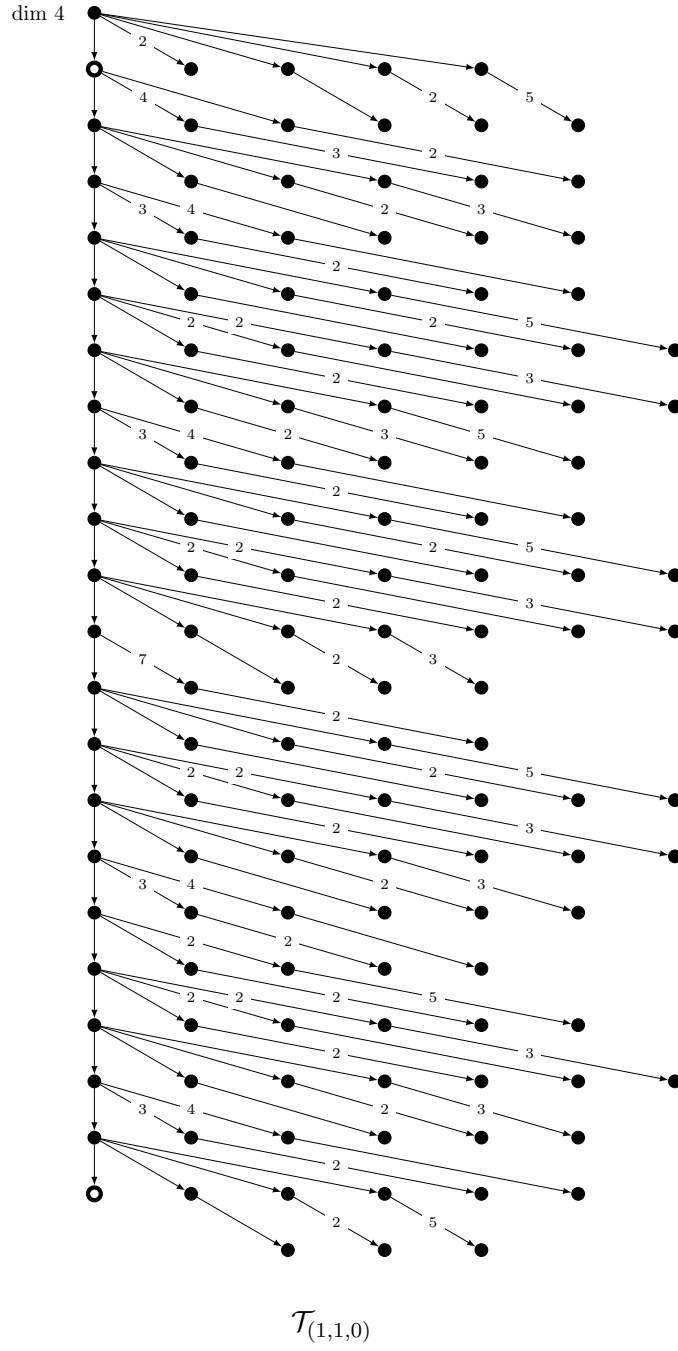
Let α denote a primitive element of \mathbb{F}_4 with $\alpha^2 \neq 1$.

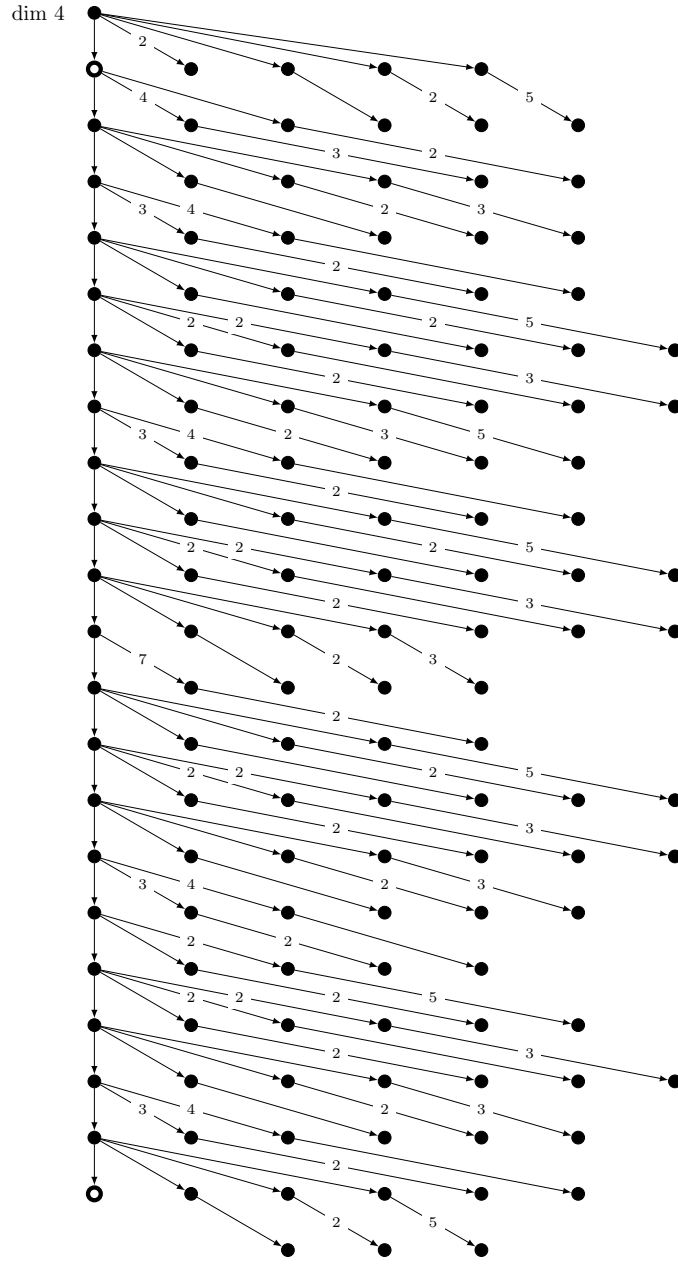




$\mathcal{T}_{(1,1,0)}$


$$\mathcal{T}_{(0,0,0)}$$

$$\mathcal{T}_{(0,0,1)}$$

$$\mathcal{T}_{(0,1,1)}$$

$$\mathcal{T}_{(0,1,0)}, \mathcal{T}_{(1,0,0)}, \mathcal{T}_{(1,2,0)}, \mathcal{T}_{(1,3,0)}$$





$\mathcal{T}_{(1,4,0)}$

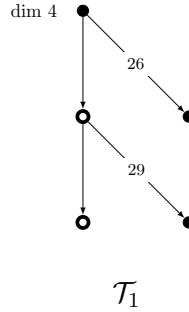
6.3 The coclass graph $\mathcal{G}_{\mathbb{F}_2}(3)$

In this section we present the maximal coclass trees of $\mathcal{G}_{\mathbb{F}_2}(3)$. There are 39 of these trees and the pro-nilpotent \mathbb{F} -algebras describing the infinite paths of these maximal coclass trees have presentations on four generators t, a, b, c , where t corresponds to a generator of the infinite subalgebra that is isomorphic to $\mathbb{F}_o[[t]]$. We only give the non-trivial relators, i.e. all products of two generators $b_1 b_2$, (except t^2 of course) that do not appear in the presentations yield a relator $b_1 b_2$. We denote the pro-nilpotent \mathbb{F}_2 of coclass by $\mathcal{A}_1, \dots, \mathcal{A}_{39}$ and denote the corresponding maximal coclass trees by $\mathcal{T}_1, \dots, \mathcal{T}_{39}$. Note that \mathcal{A}_5 and \mathcal{A}_7 are the algebras of Example 4.7 that have a common quotient of coclass 3. Hence the roots of the maximal coclass trees \mathcal{T}_5 and \mathcal{T}_7 lie in dimension 6 instead of dimension 5.

We can group the pro-nilpotent \mathbb{F}_2 -algebras of coclass 3 into different types, namely annihilator extensions of the pro-nilpotent \mathbb{F}_2 -algebras of coclass $i \in \{0, 1, 2\}$ by the vector spaces \mathbb{F}_2^{3-i} , respectively, considered as trivial modules. Recall from Section 3.3 that there is a unique isomorphism type of pro-nilpotent \mathbb{F}_2 -algebras of coclass 0, a unique isomorphism type of coclass 1 and there are six isomorphism types of coclass 2.

Annihilator extensions of $\mathbb{F}_o[[t]]$ by \mathbb{F}_2^3 :

$$\mathcal{A}_1 = \langle t, a, b, c \mid \dots \rangle$$



Annihilator extensions of $\langle t, a \mid at, ta, a^2 \rangle$ by \mathbb{F}_2^2 :

$$\mathcal{A}_2 = \langle t, a, b, c \mid a^2 - b, \dots \rangle$$

$$\mathcal{A}_3 = \langle t, a, b, c \mid ta - b, \dots \rangle$$

$$\mathcal{A}_4 = \langle t, a, b, c \mid a^2 - b, ta - b, \dots \rangle$$

$$\mathcal{A}_5 = \langle t, a, b, c \mid a^2 - c, ta - b, \dots \rangle$$

$$\mathcal{A}_6 = \langle t, a, b, c \mid at - b, \dots \rangle$$

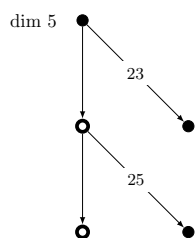
$$\mathcal{A}_7 = \langle t, a, b, c \mid a^2 - c, at - b, \dots \rangle$$

$$\mathcal{A}_8 = \langle t, a, b, c \mid ta - b, at - b, \dots \rangle$$

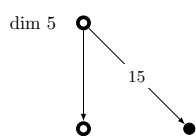
$$\mathcal{A}_9 = \langle t, a, b, c \mid a^2 - c, ta - b, at - b, \dots \rangle$$

$$\mathcal{A}_{10} = \langle t, a, b, c \mid ta - c, at - b, \dots \rangle$$

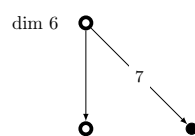
$$\mathcal{A}_{11} = \langle t, a, b, c \mid a^2 - b - c, ta - c, at - c, \dots \rangle$$



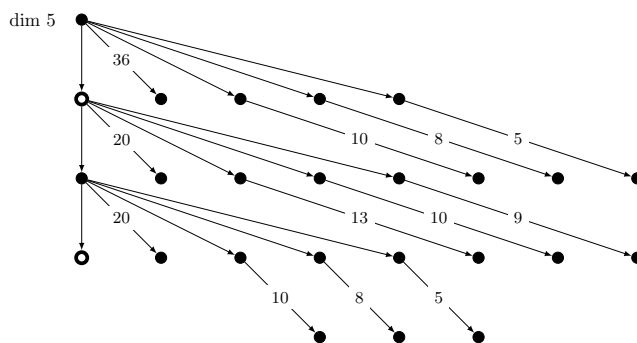
$\mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_6$



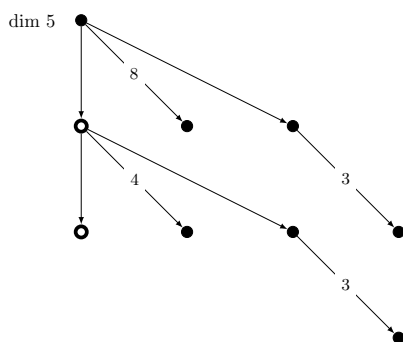
\mathcal{T}_4



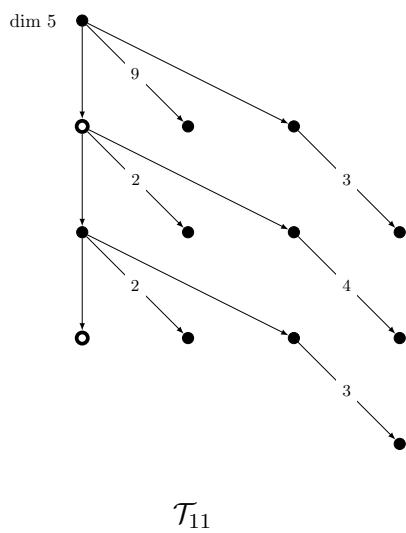
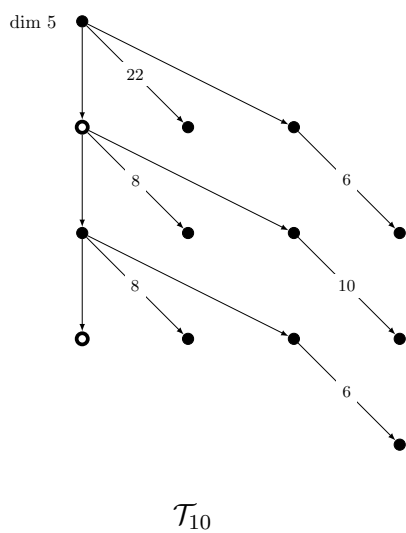
$\mathcal{T}_5, \mathcal{T}_7$



\mathcal{T}_8



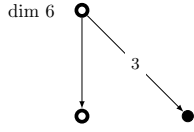
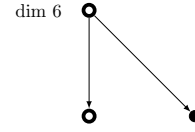
\mathcal{T}_9



Annihilator extensions of $\langle t, a, b \mid at, ta, a^2 - b, bt, tb, b^2, ab, ba \rangle$ by \mathbb{F}_2 :

$$\mathcal{A}_{12} = \langle t, a, b, c \mid a^2 - b, ab - c, ba - c, \dots \rangle$$

$$\mathcal{A}_{13} = \langle t, a, b, c \mid a^2 - b, ab - c, ba - c, ta - c, \dots \rangle$$

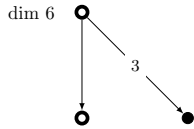
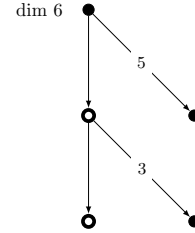

 \mathcal{T}_{12}

 \mathcal{T}_{13}

Annihilator extensions of $\langle t, a, b \mid at, ta - b, a^2, bt, tb, b^2, ab, ba \rangle$ by \mathbb{F}_2 :

$$\mathcal{A}_{14} = \langle t, a, b, c \mid ta - b, tb - c, \dots \rangle$$

$$\mathcal{A}_{15} = \langle t, a, b, c \mid ta - b, tb - c, a^2 - c, \dots \rangle$$

$$\mathcal{A}_{16} = \langle t, a, b, c \mid ta - b, tb - c, at - c, \dots \rangle$$

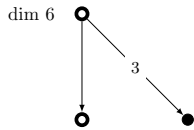
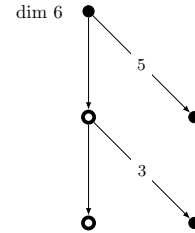

 $\mathcal{T}_{14}, \mathcal{T}_{16}$

 \mathcal{T}_{15}

Annihilator extensions of $\langle t, a, b \mid at - b, ta, a^2, bt, tb, b^2, ab, ba \rangle$ by \mathbb{F}_2 :

$$\mathcal{A}_{17} = \langle t, a, b, c \mid at - b, bt - c, \dots \rangle$$

$$\mathcal{A}_{18} = \langle t, a, b, c \mid at - b, a^2 - c, \dots \rangle$$

$$\mathcal{A}_{19} = \langle t, a, b, c \mid at - b, bt - c, ta - c, \dots \rangle$$


 $\mathcal{T}_{17}, \mathcal{T}_{19}$

 \mathcal{T}_{18}

Annihilator extensions of $\langle t, a, b \mid at, ta - b, a^2 - b, bt, tb, b^2, ab, ba \rangle$ by \mathbb{F}_2 :
none

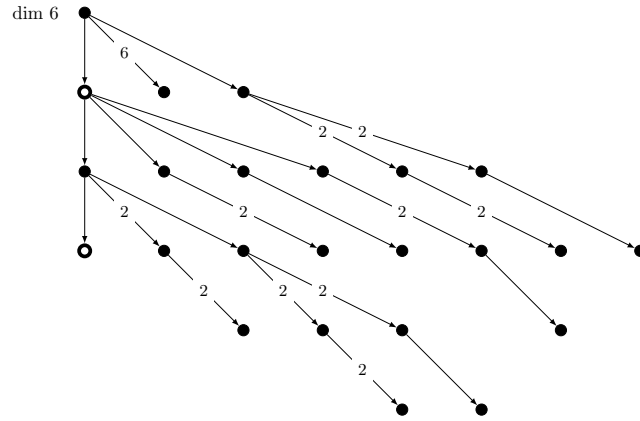
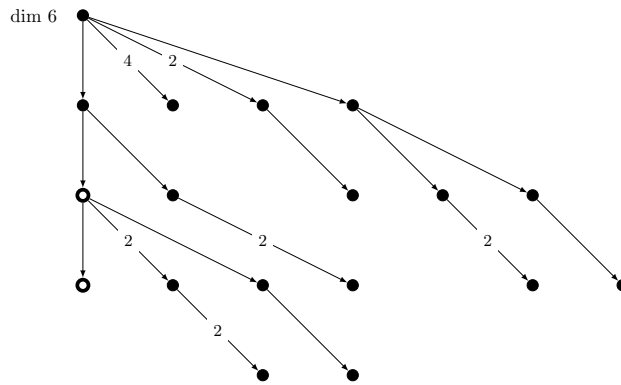
Annihilator extensions of $\langle t, a, b \mid at - b, ta - b, a^2, bt, tb, b^2, ab, ba \rangle$ by \mathbb{F}_2 :

$$\mathcal{A}_{20} = \langle t, a, b, c \mid at - b - c, ta - b, tb - c, bt - c, \dots \rangle$$

$$\mathcal{A}_{21} = \langle t, a, b, c \mid at - b, ta - b, tb - c, bt - c, a^2 - c, \dots \rangle$$

$$\mathcal{A}_{22} = \langle t, a, b, c \mid at - b - c, ta - b, tb - c, bt - c, a^2 - c, \dots \rangle$$

$$\mathcal{A}_{23} = \langle t, a, b, c \mid at - b, ta - b, tb - c, bt - c, \dots \rangle$$


 \mathcal{T}_{20}

 \mathcal{T}_{21}



Annihilator extensions of $\langle t, a, b \mid at, ta, a^2, bt, tb, b^2, ab, ba \rangle$ by \mathbb{F}_2 :

$$\mathcal{A}_{24} = \langle t, a, b, c \mid ab - c, \dots \rangle$$

$$\mathcal{A}_{25} = \langle t, a, b, c \mid ab - c, ba - c, \dots \rangle$$

$$\mathcal{A}_{26} = \langle t, a, b, c \mid a^2 - c, ab - c, ba - c, \dots \rangle$$

$$\mathcal{A}_{27} = \langle t, a, b, c \mid a^2 - c, ab - c, b^2 - c, \dots \rangle$$

$$\mathcal{A}_{28} = \langle t, a, b, c \mid ab - c, ta - c, \dots \rangle$$

$$\mathcal{A}_{29} = \langle t, a, b, c \mid ab - c, ba - c, ta - c, \dots \rangle$$

$$\mathcal{A}_{30} = \langle t, a, b, c \mid a^2 - c, ab - c, ba - c, ta - c, \dots \rangle$$

$$\mathcal{A}_{31} = \langle t, a, b, c \mid b^2 - c, ta - c, \dots \rangle$$

$$\mathcal{A}_{32} = \langle t, a, b, c \mid a^2 - c, b^2 - c, ta - c, \dots \rangle$$

$$\mathcal{A}_{33} = \langle t, a, b, c \mid a^2 - c, ab - c, b^2 - c, ta - c, \dots \rangle$$

$$\mathcal{A}_{34} = \langle t, a, b, c \mid ba - c, at - c, \dots \rangle$$

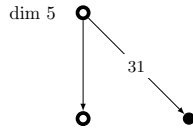
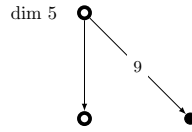
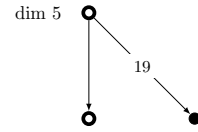
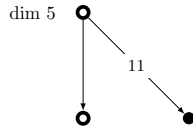
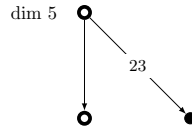
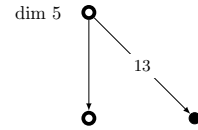
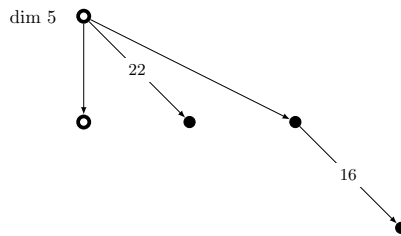
$$\mathcal{A}_{35} = \langle t, a, b, c \mid b^2 - c, at - c, \dots \rangle$$

$$\mathcal{A}_{36} = \langle t, a, b, c \mid b^2 - c, ta - c, at - c, \dots \rangle$$

$$\mathcal{A}_{37} = \langle t, a, b, c \mid ab - c, b^2 - c, ta - c, at - c, \dots \rangle$$

$$\mathcal{A}_{38} = \langle t, a, b, c \mid ta - c, tb - c, \dots \rangle$$

$$\mathcal{A}_{39} = \langle t, a, b, c \mid a^2 - c, ab - c, ba - c, b^2 - c, at - c, tb - c, \dots \rangle$$


 $\mathcal{T}_{24}, \mathcal{T}_{28}, \mathcal{T}_{32}, \mathcal{T}_{33}, \mathcal{T}_{34}, \mathcal{T}_{37}$

 \mathcal{T}_{25}

 $\mathcal{T}_{26}, \mathcal{T}_{29}, \mathcal{T}_{30}$

 \mathcal{T}_{27}

 $\mathcal{T}_{31}, \mathcal{T}_{35}$

 $\mathcal{T}_{36}, \mathcal{T}_{39}$

 \mathcal{T}_{38}

Chapter 7

Summary

7.1 Summary

Coclass theory for finite p -groups was introduced by Leedham-Green & Newman in 1980 [LGN80] and has become a very fruitful approach in the investigation of finite p -groups yielding many interesting results.

The central aim of this thesis is to initiate a coclass theory for nilpotent associative algebras over fields and hence to gain further insight into their structure. Some results of this thesis are also contained in our papers [EM15b] and [EM16]. An essential tool in our investigation are the coclass graphs $\mathcal{G}_{\mathbb{F}}(r)$ associated to the nilpotent associative \mathbb{F} -algebras of a fixed coclass r . We consider several important features of these graphs.

Infinite paths in coclass graphs

The first main result is the following theorem. It gives a complete structure description of the inverse limits associated to the infinite paths in the coclass graphs $\mathcal{G}_{\mathbb{F}}(r)$.

Theorem.

Let \mathbb{F} be an arbitrary field and let r be a non-negative integer.

- a) An associative \mathbb{F} -algebra A is isomorphic to an inverse limit of an infinite path in $\mathcal{G}_{\mathbb{F}}(r)$ if and only if it is a split extension $A = T \ltimes \text{Ann}_*(A)$ with $\dim(\text{Ann}_*(A)) = r$ and T is a subalgebra of A such that $T \cong \mathbb{F}_o[[t]]$.*
- b) The number of equivalence classes of infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$ is finite if and only if $r \leq 1$ or \mathbb{F} is finite.*

This structure description is very useful for further investigations. In particular we use it to develop a method for explicitly constructing isomorphism type representatives of the infinite-dimensional \mathbb{F} -algebras describing the inverse limits associated to an infinite path in $\mathcal{G}_{\mathbb{F}}(r)$.

Trees in coclass graphs - general case

A first description of the overall structure of $\mathcal{G}_{\mathbb{F}}(r)$ is given in the following theorem.

Theorem.

Let \mathbb{F} be an arbitrary field and let r be a non-negative integer. Then $\mathcal{G}_{\mathbb{F}}(r)$ is the disjoint union of its maximal descendant trees. Let R be a root of a maximal descendant tree in $\mathcal{G}_{\mathbb{F}}(r)$. If $r = 0$, then $\dim(R) = 1$. If $r \geq 1$, then $r + 1 \leq \dim(R) \leq 2r$ holds and these bounds are sharp.

The maximal descendant trees of $\mathcal{G}_{\mathbb{F}}(r)$ can contain either zero, one or several infinite paths starting at their roots. A coclass tree is defined to contain exactly one infinite path starting at its root. Coclass trees correspond one-to-one to equivalence classes of infinite paths. We prove the following surprising result.

Theorem.

Let \mathbb{F} be an arbitrary field and let r be a non-negative integer. Let \mathcal{T} be a maximal coclass tree in $\mathcal{G}_{\mathbb{F}}(r)$. Then \mathcal{T} has bounded depth, i.e. there is a non-negative integer c , such that for every algebra in \mathcal{T} the distance to an algebra on the unique infinite path starting at the root of \mathcal{T} is at most c .

Trees in coclass graphs - finite field case

Let \mathbb{F} be a finite field and r be a non-negative integer. Then the coclass graphs $\mathcal{G}_{\mathbb{F}}(r)$ consist of finitely many maximal descendant trees. We can further strengthen this result to the following theorem.

Theorem.

Let \mathbb{F} be an arbitrary finite field and r be a non-negative integer. Then $\mathcal{G}_{\mathbb{F}}(r)$ is the disjoint union of finitely many maximal coclass trees and finitely many other vertices.

This makes coclass graphs over finite fields accessible for computational experiments using the algorithms we exhibited in the thesis. We compute various finite parts of coclass graphs for small fields and small coclasses. The striking observation in the experimental data is that all coclass trees in our examples exhibit a periodic pattern. Based on this observation, we propose the following conjecture for coclass graphs over finite fields.

Conjecture.

Let \mathbb{F} be an arbitrary finite field and let r be a non-negative integer. Furthermore, let \mathcal{T} be a maximal coclass tree in $\mathcal{G}_{\mathbb{F}}(r)$. Then \mathcal{T} is virtually periodic and the algebras in \mathcal{T} can be described by finitely many parametrized presentations.

The conjecture holds for $r = 0$ and we give a proof for $r = 1$. For coclass 2 we give a very detailed conjecture describing the structure of the coclass graphs.

Final comments

Our results concerning the structure and number of infinite paths in coclass graphs can be considered as analogues of Theorems C and D from coclass theory for finite p -groups. The results concerning the depth of coclass trees and the conjectured periodicity indicate that from a coclass theoretic viewpoint the nilpotent associative \mathbb{F} -algebras over finite fields behave

similar to 2-groups. Note that the proofs in the theory for nilpotent associative \mathbb{F} -algebras are completely different than the proofs in the p -group case and often much simpler.

7.2 Zusammenfassung

Koklassentheorie für endliche p -Gruppen wurde 1980 von Leedham-Green & Newman [LGN80] eingeführt. Sie ist seitdem zu einem sehr erfolgreichen Ansatz in der Untersuchung von p -Gruppen geworden, der viele interessante Resultate hervorgebracht hat.

Das Hauptziel dieser Arbeit ist es eine Koklassentheorie für nilpotente assoziative Algebren über Körpern zu entwickeln und somit weitere Einblicke in deren Struktur zu bekommen. Einige Ergebnisse dieser Arbeit sind auch in unseren Veröffentlichungen [EM15b] und [EM16] enthalten. Ein wichtiges Werkzeug sind die Koklassengraphen $\mathcal{G}_{\mathbb{F}}(r)$, die zu den assoziativen \mathbb{F} -Algebren einer festen Koklasse r gehören. Wir betrachten verschiedene wichtige Eigenschaften dieser Graphen.

Unendliche Pfade in Koklassengraphen

Das erste Hauptresultat ist das folgende Theorem. Wir geben eine vollständige Strukturbeschreibung für die inversen Limes, die zu unendlichen Pfaden in den Koklassengraphen $\mathcal{G}_{\mathbb{F}}(r)$ gehören.

Theorem.

Sei \mathbb{F} ein beliebiger Körper und r eine nichtnegative ganze Zahl.

- a) Eine assoziative \mathbb{F} -Algebra A ist genau dann isomorph zu einem inversen Limes eines unendlichen Pfades in $\mathcal{G}_{\mathbb{F}}(r)$, wenn sie eine zerfallende Erweiterung $A = T \ltimes \text{Ann}_*(A)$ mit $\dim(\text{Ann}_*(A)) = r$ ist. Hierbei ist T eine Unteralgebra von A , so dass $T \cong \mathbb{F}_\circ[[t]]$.*
- b) Die Anzahl der Äquivalenzklassen von unendlichen Pfaden in $\mathcal{G}_{\mathbb{F}}(r)$ ist endlich genau dann, wenn $r \leq 1$ oder \mathbb{F} ein endlicher Körper ist.*

Diese Strukturbeschreibung ist sehr hilfreich für weitere Untersuchungen. Insbesondere benutzen wir sie um eine Methode zu entwickeln, mit der wir die unendlichdimensionalen \mathbb{F} -Algebren, die zu unendlichen Pfaden in $\mathcal{G}_{\mathbb{F}}(r)$ gehören, bis auf Isomorphie konstruieren können.

Bäume in Koklassengraphen - Allgemeiner Fall

Eine erste Beschreibung der Gesamtstruktur von $\mathcal{G}_{\mathbb{F}}(r)$ ist durch das folgende Theorem gegeben.

Theorem.

Sei \mathbb{F} ein beliebiger Körper und r eine nichtnegative ganze Zahl. Dann ist $\mathcal{G}_{\mathbb{F}}(r)$ die disjunkte Vereinigung seiner maximalen Nachfolgerbäume. Sei R die Wurzel eines maximalen Nachfolgerbaums in $\mathcal{G}_{\mathbb{F}}(r)$. Für $r = 0$ gilt $\dim(R) = 1$. Für $r \geq 1$ gilt $r + 1 \leq \dim(R) \leq 2r$ und diese Schranken sind bestmöglich.

Die maximalen Nachfolgerbäume von $\mathcal{G}_{\mathbb{F}}(r)$ können entweder keinen, einen oder mehrere unendliche, an den Wurzeln der Bäume beginnende, Pfade enthalten. Ein Koklassenbaum

ist so definiert, dass er genau einen unendlichen, an der Wurzel beginnenden, Pfad enthält. Koklassenhume korrespondieren eins-zu-eins zu Aquivalenzklassen von unendlichen Pfaden. Wir beweisen das uberraschende Resultat, dass fur einen beliebigen Korper alle maximalen Koklassenhume beschrankte Tiefe haben.

Theorem.

Sei \mathbb{F} ein beliebiger Korper und r eine nichtnegative ganze Zahl. Sei \mathcal{T} ein maximaler Koklassenhum in $\mathcal{G}_{\mathbb{F}}(r)$. Dann hat \mathcal{T} beschrankte Tiefe, d.h. es gibt eine nichtnegative Zahl c , so dass fur jede Algebra in \mathcal{T} der Abstand zu einer Algebra auf dem eindeutigen unendlichen Pfad, der an der Wurzel von \mathcal{T} beginnt, hochstens c ist.

Bume in Koklassengraphen - Endliche Korper

Sei \mathbb{F} ein beliebiger endlicher Korper und r eine nichtnegative ganze Zahl. Dann besteht der Koklassengraph $\mathcal{G}_{\mathbb{F}}(r)$ aus endlich vielen maximalen Nachfolgerbumen. Wirverstarken dieses Resultat im folgenden Theorem.

Theorem.

Sei \mathbb{F} ein beliebiger endlicher Korper und r eine nichtnegative ganze Zahl. Dann ist $\mathcal{G}_{\mathbb{F}}(r)$ die disjunkte Vereinigung von endlich vielen maximalen Koklassenhumen und endlich vielen anderen Knoten.

Das macht Koklassengraphen uber endlichen Korpern zuganglich fur Computerexperimente mit Hilfe der in der Arbeit entwickelten Algorithmen. Wir berechnen verschiedene endliche Stucke von Koklassengraphen fur kleine Korper und kleine Koklassen. In den experimentellen Daten lasst sich bemerkenswerterweise in allen berechneten Koklassenhumen ein periodisches Verhalten erkennen. Auf Basis dieser Beobachtung stellen wir die folgende Vermutung auf.

Vermutung.

Sei \mathbb{F} ein beliebiger endlicher Korper und r eine nichtnegative ganze Zahl. Weiter sei \mathcal{T} ein maximaler Koklassenhum in $\mathcal{G}_{\mathbb{F}}(r)$. Dann ist \mathcal{T} virtuell periodisch und die Algebren in \mathcal{T} lassen sich durch endlich viele parametrisierte Präsentationen beschreiben.

Diese Vermutung ist wahr fur $r = 0$ und wir geben auch einen Beweis fur $r = 1$. Fur Koklasse 2 geben wir eine sehr detaillierte Vermutung fur die Struktur der Koklassengraphen.

Letzte Kommentare

Unsere Resultate, welche die Struktur und Anzahl der unendlichen Pfade beschreiben, konnen als Analoga zu Theorem C und D aus der Koklassentheorie fur endliche p -Gruppen gesehen werden. Das Ergebnis zur Tiefe von Koklassengraphen und die vermutete Periodizitat weisen darauf hin, dass aus koklassentheoretischer Sicht nilpotente assoziative \mathbb{F} -Algebren uber endlichen Korpern ein ahnliches Verhalten zeigen wie 2-Gruppen. Man beachte, dass die Beweise in der Theorie fur nilpotente assoziative \mathbb{F} -Algebren sich vollstandig von denen im Fall der p -Gruppen unterscheiden und oftmals einfacher sind.

Acknowledgements

First of all, I would like to thank my supervisor Prof. Dr. Bettina Eick for making this thesis possible through her guidance and many stimulating discussions. I would also like to thank her for the diverse opportunities to visit conferences and workshops within and outside of Germany.

I also want to thank my second referee Prof. Dr. Andrea Caranti for many helpful remarks and of course for making the trip from Trento to Braunschweig to attend my defense.

I would like to thank my friends and colleagues here at the TU Braunschweig for an amazing time with lots of interesting discussions while at work and also for the necessary distractions outside of our offices.

Very warm thanks go to my family for always supporting me during the whole time of my studies. Furthermore, I would like to thank all of my friends outside of university for their continuous support.

I want to thank the DFG for the support within the DFG Priority Project SPP 1489, which made it possible for me to work on this project. Additionally, I want to thank the DAAD for the possibility of a research visit at Monash University, Melbourne, Australia.

Bibliography

- [Ber11] C. Bertram. Ein Algorithmus zur Klassifikation nilpotenter Algebren. Master's thesis, TU Braunschweig, 2011.
- [Bla58] N. Blackburn. On a special class of p -groups. *Acta Math.*, 100:45–92, 1958.
- [CMN97] A. Caranti, S. Mattarei, and M. F. Newman. Graded Lie algebras of maximal class. *Trans. Amer. Math. Soc.*, 349(10):4021–4051, 1997.
- [Cou14] M. Couson. Character degrees of finite p -groups by coclass. *J. Algebra*, 418:91–109, 2014.
- [CVL00] A. Caranti and M. R. Vaughan-Lee. Graded Lie algebras of maximal class. IV. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 29(2):269–312, 2000.
- [DE13] A Distler and B. Eick. Coclass theory for nilpotent semigroups via their associated algebras. *J. Algebra*, 373:491 – 501, 2013.
- [dG10] W. A. de Graaf. Classification of nilpotent associative algebras of small dimension. *arXiv:1009.5339*, 2010.
- [Die10] H. Dietrich. A new pattern in the graph of p -groups of maximal class. *Bull. London Math. Soc.*, 42(6):1073–1088, 2010.
- [dS00] M. du Sautoy. Counting p -groups and nilpotent groups. *Inst. Hautes Études Sci. Publ. Math.*, (92):63–112 (2001), 2000.
- [EF11] B. Eick and D. Feichtenschlager. Computation of low-dimensional (co)homology groups for infinite sequences of p -groups with fixed coclass. *Internat. J. Algebra Comput.*, 21(4):635–649, 2011.
- [Eic06] B. Eick. Automorphism groups of 2-groups. *J. Algebra*, 300(1):91 – 101, 2006. Computational Algebra Special Issue Celebrating the 65th birthday of Charles Leedham-Green.
- [Eic08] B. Eick. Computing automorphism groups and testing isomorphism for modular group algebras. *J. Algebra*, 320(11):3895–3910, 2008.
- [Eic16] B. Eick. Modisom 2.3.3 - a GAP4 package. 2016.
- [ELG08] B. Eick and C.R. Leedham-Green. On the classification of prime-power groups by coclass. *Bull. London Math. Soc.*, 40(2):274–288, 2008.

- [ELGNO13] B. Eick, C. R. Leedham-Green, M. F. Newman, and E. A. O'Brien. On the classification of groups of prime-power order by coclass: The 3-groups of coclass 2. *Internat. J. Algebra and Comput.*, 23(05):1243–1288, 2013.
- [EM15a] B. Eick and T. Moede. ccalgs version 1.0 - a GAP4 package, available at <http://www.icm.tu-bs.de/~tobmoede/ccalgs/>. 2015.
- [EM15b] B. Eick and T. Moede. Nilpotent associative algebras and coclass theory. *J. Algebra*, 434:249–260, 2015.
- [EM16] B. Eick and T. Moede. Coclass theory for finite nilpotent associative algebras: Algorithms and a periodicity conjecture. *accepted by Exp. Math.*, 2016.
- [EO99] B. Eick and E. A. O'Brien. Enumerating p -groups. *J. Austral. Math. Soc. Ser. A*, 67(2):191–205, 1999. Group theory.
- [FPR07] J. Farina and C. Pendergrass-Rice. A few properties of just infinite algebras. *Comm. Algebra*, 35(5):1703–1707, 2007.
- [GAP15] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.7.9*, 2015.
- [HB12] B. Huppert and N. Blackburn. *Finite Groups II*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2012.
- [Jen41] S. A. Jennings. The structure of the group ring of a p -group over a modular field. *Trans. Amer. Math. Soc.*, 50(1):175–185, 1941.
- [KP69] R. Kruse and D. Price. *Nilpotent Rings*. Gordon and Breach, New York, 1969.
- [Lan71] S.M. Lane. *Categories for the working mathematician*. Graduate texts in mathematics. Springer-Verlag, 1971.
- [LG94] C. R. Leedham-Green. The structure of finite p -groups. *J. London Math. Soc.*, 2(50):49–67, 1994.
- [LGM02] C. R. Leedham-Green and S. McKay. *The structure of groups of prime-power order*. London Math. Soc. Monographs. Oxford Science Publications, 2002.
- [LGN80] C. R. Leedham-Green and M. F. Newman. Space groups and groups of prime-power order. I. *Arch. Math. (Basel)*, 35(3):193–202, 1980.
- [NO99] M. F. Newman and E. A. O'Brien. Classifying 2-groups by coclass. *Trans. Amer. Math. Soc.*, 351(1):131–169, 1999.
- [Pie82] R.S. Pierce. *Associative algebras*. Graduate texts in mathematics. Springer-Verlag, 1982.
- [RS94] D.M. Riley and J.F. Semple. The coclass of a restricted Lie algebra. *Bull. London Math. Soc.*, 26(5):431–437, 1994.

- [Sha94] A. Shalev. The structure of finite p -groups: effective proof of the coclass conjectures. *Invent. Math.*, 115(2):315–345, 1994.
- [SZ97] A. Shalev and E. Zelmanov. Narrow Lie algebras: A coclass theory and a characterization of the Witt algebra. *J. Algebra*, 189(2):294 – 331, 1997.
- [Wed08] J. H. MacLagan Wedderburn. On hypercomplex numbers. *Proc. London Math. Soc.*, s2-6(1):77–118, 1908.